Time

To study this unit will take you about 10 hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you have planned to cover the topic.

Section A: Straight lines

Section A1: Gradients of straight lines

Plotting points that satisfy a relationship of the form $y = ax + b$ and joining the points plotted will ALWAYS give a straight line.

Equations of the form $y = ax + b$ are equations of straight lines (but not all straight lines have equations of the form $y = ax + b$). They are called **linear** equations. The word **linear** contains the word **line**.

To plot the graph of the line with equation $y = ax + b$ you:

(i) find and tabulate the coordinates of three points on the line:

(ii) plot the points on a labelled coordinate grid. Use X (two small crossing lines) to indicate the position of the point.

(iii) join the points, extending the line at both sides.

(iv) label the line with its equation.

The following are elements for inclusion in a worksheet for pupils to develop the concept of gradient. Pupils are expected to work in groups and discuss their work with each other.

An equivalent term for ‘gradient’ is ‘slope’, common in North America and in textbooks from there.

Worksheet 4 (outline)

Objective: The pupils should relate ‘steepness’ and gradient (‘stepsize’ when moving 1 unit across) and be able to find gradient of lines (positive gradients only).

1. a) Using the four steps outlined draw on the same coordinate grid lines with the following equations
   (i) $y = x$ (ii) $y = 2x$ (iii) $y = 3x$ (iv) $y = 4x$

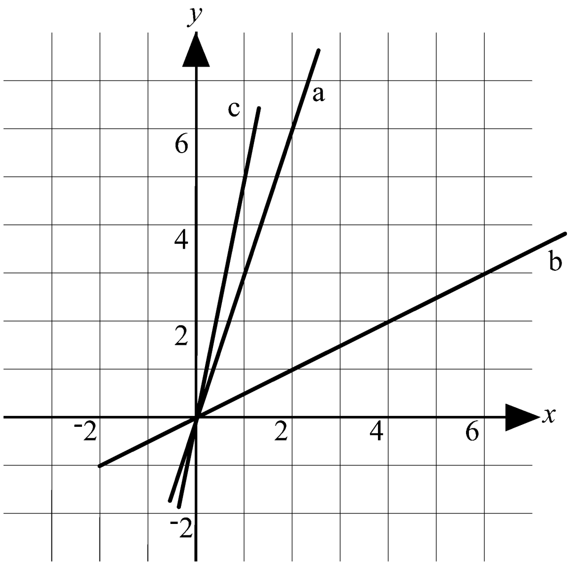
   b) Which line is the steepest?

   c) Which line is least steep?

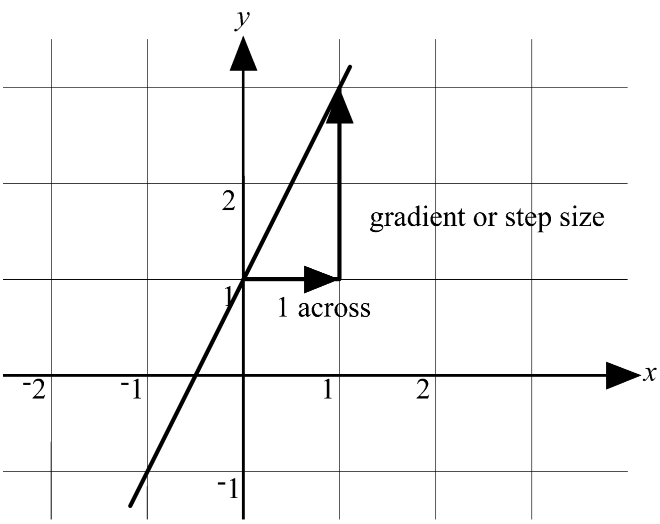
   d) Which part of the equation tells you how steep the line is?

2. Which line is the steepest in each of the following pairs?
   a) $y = 3x$ and $y = 5x$ b) $y = 4x$ and $y = x$
   c) $y = 3\frac{1}{2}x$ and $y = 3\frac{3}{4}x$ d) $y = 4.1x$ and $y = 4.11x$

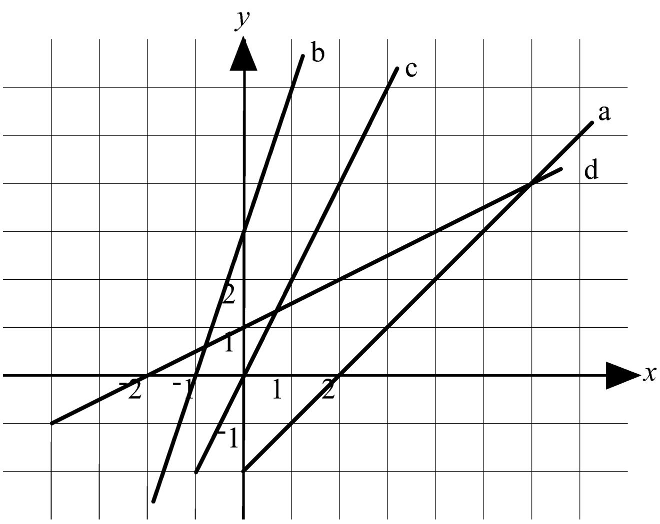
3. Which equation belongs to line a, b and c? Choose from
   $y = \frac{1}{2}x$, $y = 5x$ and $y = 3x$



The 'steepness' of a line is called its gradient, or slope, or step size. It tells you how high up the graph goes for ONE step across.



4. What is the gradient of the lines a, b, c and d?



5. a) The points P (2, 1) and Q(4, 5) have been plotted.

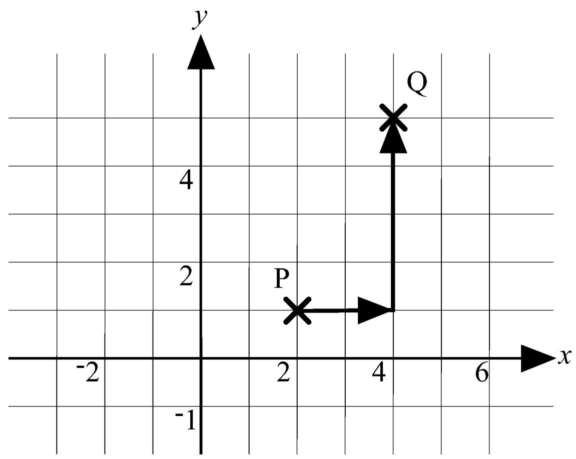
How many steps to the right and how many steps up will you move to move from P to Q?

Complete:

If you move .... steps to the right, you move .... steps up.

If you move 1 step to the right, you will move up .... steps.

What is the gradient of the line through P and Q?



b) Calculate the gradient of the lines passing through A and B if

(i) A(1, 2) and B(3, 8)

(ii) A(4, 1) and B(7, 7)

(iii) A(⁻3, ⁻2) and B(1, 10)

(iv) A(⁻4, 2) and B(⁻1, 6)

(v) A(⁻2, 4) and B(2, 8)

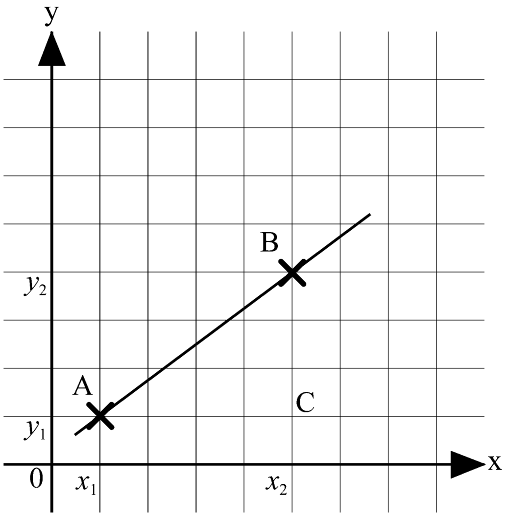
(vi) A(3, 1) and B(7, 1)

(vii) A(2, 1) and B(2, 5)

Self mark exercise 1

1. Work the suggested questions in worksheet 4.
2. Are ALL lines of the format $y = ax + b$? Justify your answer.
3. What is special about lines with gradient 0? What type of equation do these line have?
4. Which lines have NO gradient? What type of equation do these line have?
5. a) What is the gradient of the x-axis? What is its equation?

   b) What is the gradient of the y-axis? What is its equation?
6. Derive an expression for the gradient of a line through the points with coordinates $A(x_1, y_1)$ and $B(x_2, y_2)$.



   Express AC in terms of the x-coordinates of A and B, express BC in terms of the y coordinates of A and B. Hence find the gradient of AB.

7. a) Did you find in 6 that the gradient of AB is $\frac{y_2 - y_1}{x_2 - x_1}$. Explain

   (i) why you can also write for the gradient $\frac{y_1 - y_2}{x_1 - x_2}$.

   (ii) why $x_1 \neq x_2$. What line do you have in case the two x-coordinates are equal?

   b) What trigonometric ratio of angle BAC is given by the gradient?

   c) Relate the tangent of the angle between the line and the positive x-axis to the gradient of the line.

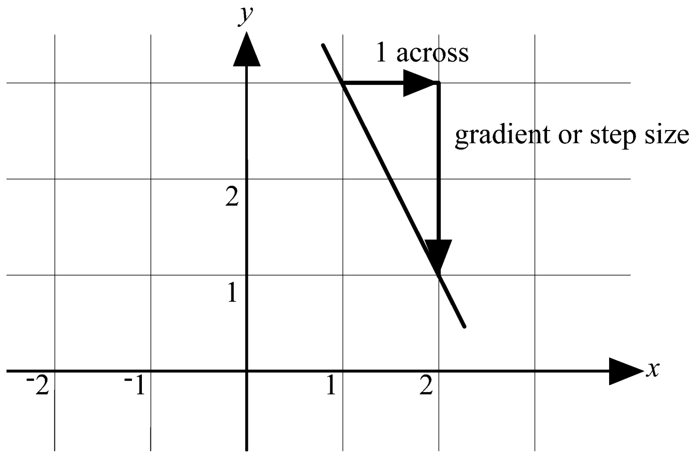
   d) Relate the size of the angle between the line and the positive x-axis (0°, acute, 90° or obtuse) to the gradient of the line AB.

Check your answers at the end of this unit

In the worksheet outlined above the work is restricted to lines with positive gradient, zero gradient (horizontal lines, with equations of the form $y = k$, where k is a real number) or no gradient (vertical lines, with equations of the form $x = p$, where p is a real number)

The next step is to introduce lines with negative gradient.

The following diagram could be used.



For one step across you have to move down to reach the line again. Lines with negative slope make an obtuse angle with the positive x-axis. Lines with positive gradients make acute angles with the positive x-axis.

Unit 2, Practice activity 1

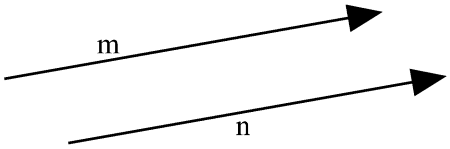
1. The suggestions in worksheet W4 need to be fine tuned to different attainment levels of pupils in the class. Describe in detail how you would do that.
2. Write a worksheet for pupils of different attainment levels to develop the concept of negative gradients. Justify the structure of your worksheet and give the expected working.
3. In self mark exercise 1 question 6 you derived a general expression for the gradient of a line through two points with given coordinates. Would you want your pupils to be able to recall and apply the ‘formula’ or would you want them to refer to a diagram when finding gradients? Justify your answer.

   How important, in general, is it for pupils to be able to recall ‘formula’ and how important is it to relate concepts to a ‘picture’ or ‘diagram’?

Present your assignment to your supervisor or study group for discussion.

Section A2: Parallel lines

If the angle between two lines in a plane is 0° the lines are called parallel. The (perpendicular) distance between the two lines is constant (and could be zero).



The lines m and n are parallel to each other. Notation used is m || n. In the diagram arrows are placed to indicate parallel lines.

The following is an outline of a worksheet that can be set to pupils to discover the relation between parallel lines and their gradients. Pupils are expected to work in groups and discuss their work with each other.

Worksheet 5 (outline)

Objective: Pupils should discover that (i) parallel lines have the same gradient (ii) if two lines have the same gradient the lines are parallel.

1. Draw the lines with the following equations using the same coordinate grid

   Remember to follow the four steps
   (i) find and tabulate the coordinates of three points on the line
   (ii) plot the points on a labelled coordinate grid. Use X (two small crossing lines) to indicate the position of the point.
   (iii) join the points, extending the line on both sides
   (iv) label the line with its equation

   $y = x$

   $y = x + 1$

   $y = x + 2$

   $y = x - 1$

   $y = x - 2$

   a) What are the gradients of the lines?

   Copy and complete:
   The lines all have as a gradient ..... and are .... to each other.

2. Draw and label the lines with the following equations on a coordinate grid

   $y = 2x$

   $y = 2x + 3$

   $y = 2x - 2$

   Copy and complete:
   The lines all have as a gradient ..... and are .... to each other.

3. Draw and label the lines with the following equations on a coordinate grid

   $y = \frac{1}{2}x$

$y = \frac{1}{2}x + 4$

$y = \frac{1}{2}x - 3$

Copy and complete:
The lines all have as a gradient ..... and are .... to each other.

4. Draw and label the lines with the following equations on a coordinate grid

   $y = {}^{-}2x$

   $y = {}^{-}2x + 1$

   $y = {}^{-}2x - 2$

   Copy and complete:
   The lines all have as a gradient ..... and are .... to each other.

5. Draw and label the lines with the following equations on a coordinate grid

   $y = 2$

   $y = 3$

   $y = {}^{-}2$

   Copy and complete:
   The lines all have as a gradient ..... and are .... to each other.

6. Draw and label the lines with the following equations on a coordinate grid

   $x = 2$

   $x = 3$

   $x = {}^{-}2$

   Copy and complete:
   The lines all have as a gradient ..... and are .... to each other.

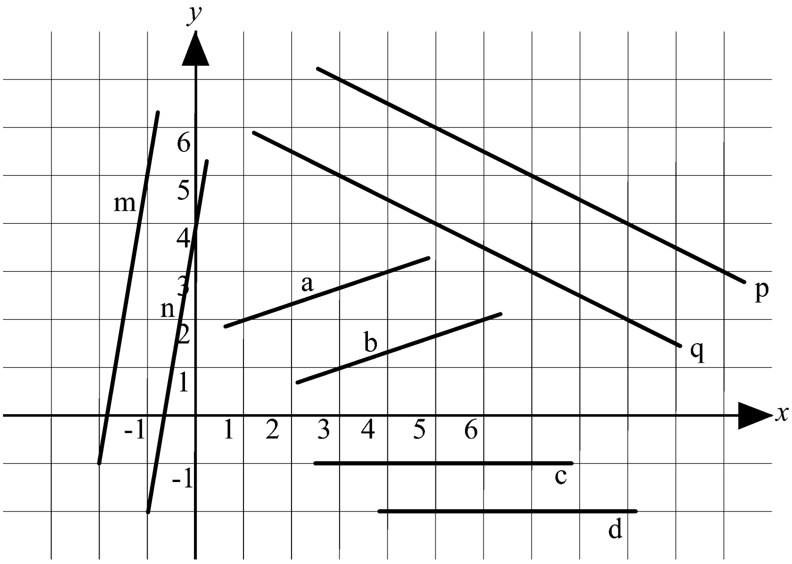
   Conclusion from question **1 - 6**:
   If two lines have an equal gradient then ....

7. Look at each set of three equations of lines.
   Which pair of lines in each set of three are parallel to each other?

a) $y = 4x - 7$	$y = 3x - 1$	$y = 4x$
b) $y = 2x + 3$	$y = 4 + 2x$	$y = 2 + 3x$
c) $y = 1 + 3x$	$y = 3 + x$	$y = 3$
d) $y = 2 + x$	$y = 2$	$y = {}^{-}2$

   e) What is the gradient of the parallel lines in a - d?

8. The diagram illustrates pairs of lines that are parallel to each other.



a) find the gradients of the lines a and b

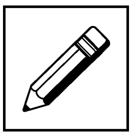
b) find the gradients of the lines m and n

c) find the gradients of the lines p and q

d) find the gradients of the lines c and d

What can you say about the gradients if two lines are parallel?

Complete the statement: If two lines are parallel then ....



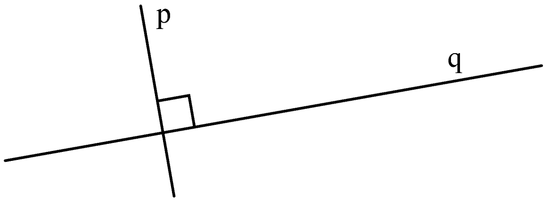
Self mark exercise 2

1. Work the questions in worksheet 5.
2. What concept is developed in questions 1 - 7 of worksheet 5?
3. What concept is developed in question 8 of worksheet 5?
4. Has an inductive or deductive method been used in worksheet 5? Explain.

Check your answers at the end of this unit

Section A3: Perpendicular lines

If the angle between two lines in a plane is 90° the lines are said to be perpendicular to each other.



The lines p and q are perpendicular to each other. The notation used is $p \perp q$. In a diagram a small square appears in the right-angle.

For parallel lines two relationships were considered
(i) If m || n then $grad_m = grad_n$ ($grad_n$ means "gradient of n")
(ii) If $grad_m = grad_n$ then m || n

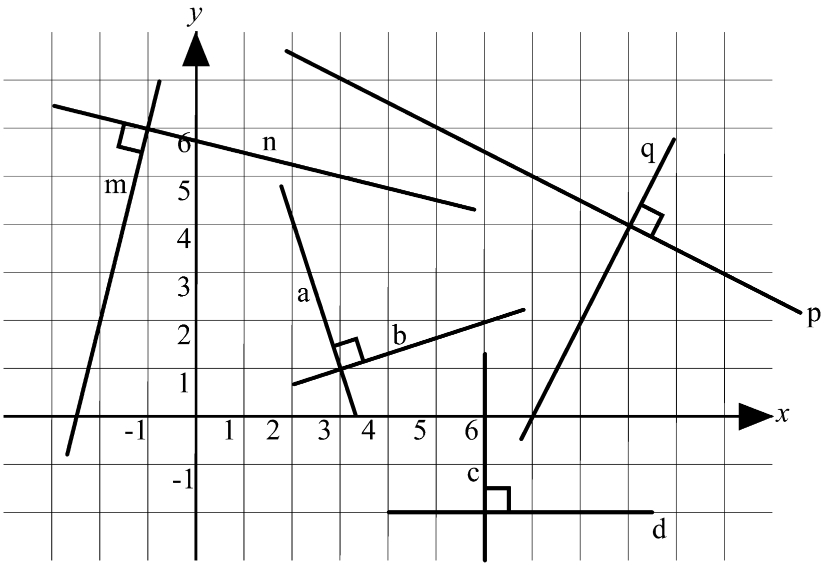
For perpendicular lines similarly two relationships are to be developed
(i) If $p \perp q$ then $grad_p \times grad_q = {}^-1$ (provided neither p or q is a horizontal or vertical line)
(ii) If $grad_p \times grad_q = {}^-1$ then $p \perp q$.

Below you find guided questions for pupils to discover the above relationships. Pupils are expected to work in groups and discuss their work with each other.

Worksheet 6 (outline)

Objective: Pupils to discover (i) if lines are perpendicular (and not parallel to the axes) the product of their gradients is $^-1$ (ii) if the product of the gradients of two lines is $^-1$ then the lines are perpendicular.

1. The diagram illustrates pairs of lines that are perpendicular to each other.



   a) Find the gradients of the lines a and b, and the product of the gradients.

b) Find the gradients of the lines m and n, and the product of the gradients.
c) Find the gradients of the lines p and q, and the product of the gradients.
d) Find the gradients of the lines c and d, and the product of the gradients.
e) What is special about the situation in d?
f) What can you say about the product of the gradients if two lines are perpendicular?

Complete the statement:

If two lines are perpendicular to each other then ...... provided ......

2. Draw and label the lines with the following equations on a coordinate grid

$y = 2$

$x = 3$

a) What are the gradients of the lines? What is the product of their gradients?
b) What is the angle the two lines make with each other at their point of intersection?

Copy and complete:

The two lines are ...... to each other, however, the product of their gradients is not ......

3. Draw the lines with the following equations using the same coordinate grid

$$y = x + 1 \text{ and } y = {}^{-}x + 3$$

a) What are the gradients of the lines? What is the product of their gradients?
b) What is the angle the two lines make with each other at their point of intersection?

Copy and complete:

The product of the gradients of the lines is ...... and the lines are ...... to each other.

4. Draw and label the lines with the following equations on a coordinate grid

$$y = {}^{-}2x + 3 \qquad y = \frac{1}{2}x - 2$$

a) What are the gradients of the lines? What is the product of their gradients?
b) What is the angle the two lines make with each other at their point of intersection?

Copy and complete:

The product of the gradients of the lines is ...... and the lines are ...... to each other.

5. Draw and label the lines with the following equations on a coordinate grid

$$y = {}^{-}3x + 6 \qquad y = \frac{1}{3}x - 4$$

a) What are the gradients of the lines? What is the product of their gradients?
b) What is the angle the two lines make with each other at their point of intersection?

Copy and complete:
The product of the gradients of the lines is ...... and the lines are ...... to each other.

6. Look at each set of three equations of lines.
Which pair of lines in each set of three are perpendicular to each other?

a) $y = 4x - 7$ $\quad y = \frac{1}{4}x - 1$ $\quad y = {}^{-}4x$

b) $y = 2x + \frac{1}{2}$ $\quad y = 4 - 2x$ $\quad y = 2 - \frac{1}{2}x$

c) $y = 3 - xy = {}^{-}3 + x$ $\quad y = \frac{1}{3}$

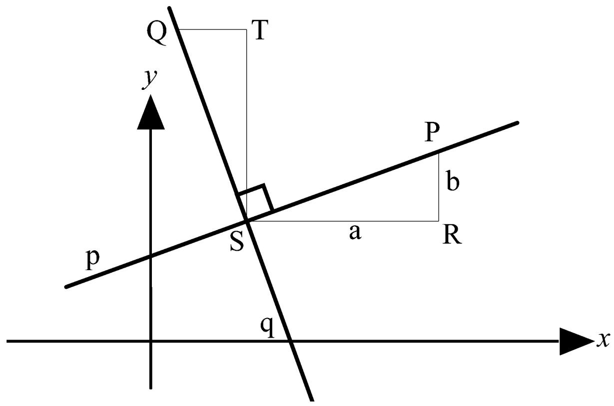
d) $y = {}^{-}2$ $\quad y = \frac{1}{2}$ $\quad x = {}^{-}4$

Self mark exercise 3

1. Work the questions in worksheet 6.
2. What concept is developed in questions 1 - 2 of worksheet 6?
3. What concept is developed in questions 3 - 6 of worksheet 6?
4. In worksheet 6 an inductive method is used. As a teacher you should be able to give a deductive proof as a few high attainers might appreciate a deductive proof.

   a) To prove:

   If $p \perp q$ then $grad_p \times grad_q = {}^-1$ (provided neither p or q is a horizontal or vertical line).
   Try to prove, deductively, the statement. If you get stuck, read the following hints.
   (i) Draw a diagram with two lines p and q perpendicular to each other.



p

   If S, the point of intersection of p and q has coordinates (x, y) and P is a point on p with coordinates $(x + a, y + b)$, find the gradient of the line p using the right-angled triangle SRP.
   As q is perpendicular to p, q is obtained by rotating p through 90°. The triangle SRP moves in this rotation to position STQ. Use the triangle (you know the lengths of the sides) to find the gradient of line q.

   Find the product of the two gradients.
   (ii) Explain the part "provided neither p or q is a horizontal or vertical line".

   b) To prove:

   If $grad_p \times grad_q = {}^-1$ then $p \perp q$.
   Try to prove, deductively, the statement. If you get stuck, read the following hints.

   (i) Let the gradient of line p be m, then the gradient of q will be $-\frac{1}{m}$ as the product is given to be ${}^-1$. Take the origin at the point of intersection of p and q. Write down the equation of p and the equation of q.

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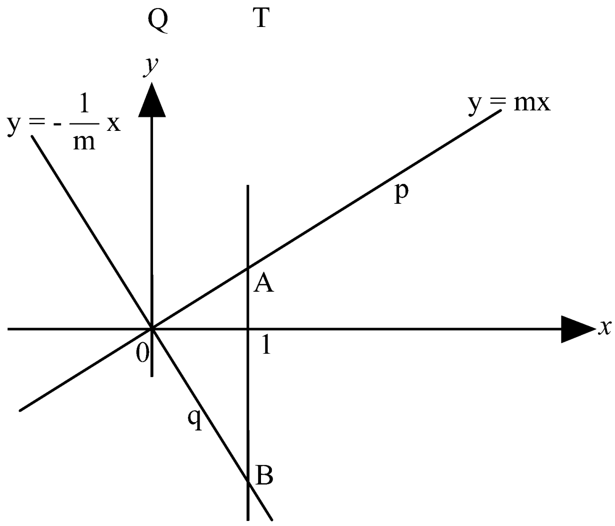
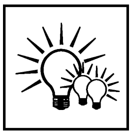


Find the coordinates of A, on p, if $x = 1$, and of B, on q, with $x = 1$.
Now consider triangle OAB. It is to be shown that angle AOB = 90°. Do this by showing that in the triangle the Pythagorean theorem holds.

Find $|OA|^2$, $|OB|^2$ and $|AB|^2$ and show that $|OA|^2 + |OB|^2 = |AB|^2$ and draw your conclusion.

Check your answers at the end of this unit.



Unit 2, Practice activity 2

1. Critically evaluate the suggestions for worksheets W5 (parallel lines) and W6 (perpendicular lines). Make adaptations appropriate for your class and prepare a lesson plan for covering the relationships as covered in worksheets 5 and 6. Carry out the lessons and write an evaluative report. Do not forget to include the lesson plan and any other material developed for the lesson.

Present your assignment to your supervisor or study group for discussion.

Section B: The equation of a straight line

Plotting corresponding values of x and y satisfying a **linear equation** of the form $y = mx + n$ will give a graph of a straight line. It includes all straight lines except vertical lines. Vertical lines have no gradient and have equations of the format $x = k$, where k is an real number.

A format covering ALL lines is $ax + by = c$, which is the **general** equation of a straight line.

The format $y = mx + n$ is frequently used as 'general' equation. This is fine when one is sure that no vertical line is involved.

Section B1: Exploring the meaning of *b* in *y = ax + b*

In section A1 you looked at a guided activity for pupils to identity the gradient with the coefficient of x in $y = ax + b$. The following outline of a worksheet is to guide pupils to interpreting the value of b in the equation $y = ax + b$ as the y-intercept.

Worksheet 7 (outline)

Objective: Pupils to discover that in the equation of a line $y = ax + b$, the value of b is the y-intercept (directed distance from O to the point of intersection of the line and the y-axis).

1. a) On the same coordinate grid draw and label the lines with equations
   $y = 2x - 4$
   $y = 2x - 2$
   $y = 2x$
   $y = 2x + 3$

   b) Copy and complete these sentences:
   The line with equation $y = 2x - 4$ crosses the y-axis at the point with y coordinate ....
   The line with equation $y = 2x - 2$ crosses the y-axis at the point with y coordinate ....
   The line with equation $y = 2x$ crosses the y-axis at the point with y coordinate ....
   The line with equation $y = 2x + 3$ crosses the y-axis at the point with y coordinate ....

   c) Where will the line with equation $y = 2x + 9$ cross the y-axis?

   d) Where will the line with equation $y = 2x - 56$ cross the y-axis?

   e) Where will the line with equation $y = 2x + p$ cross the y-axis?

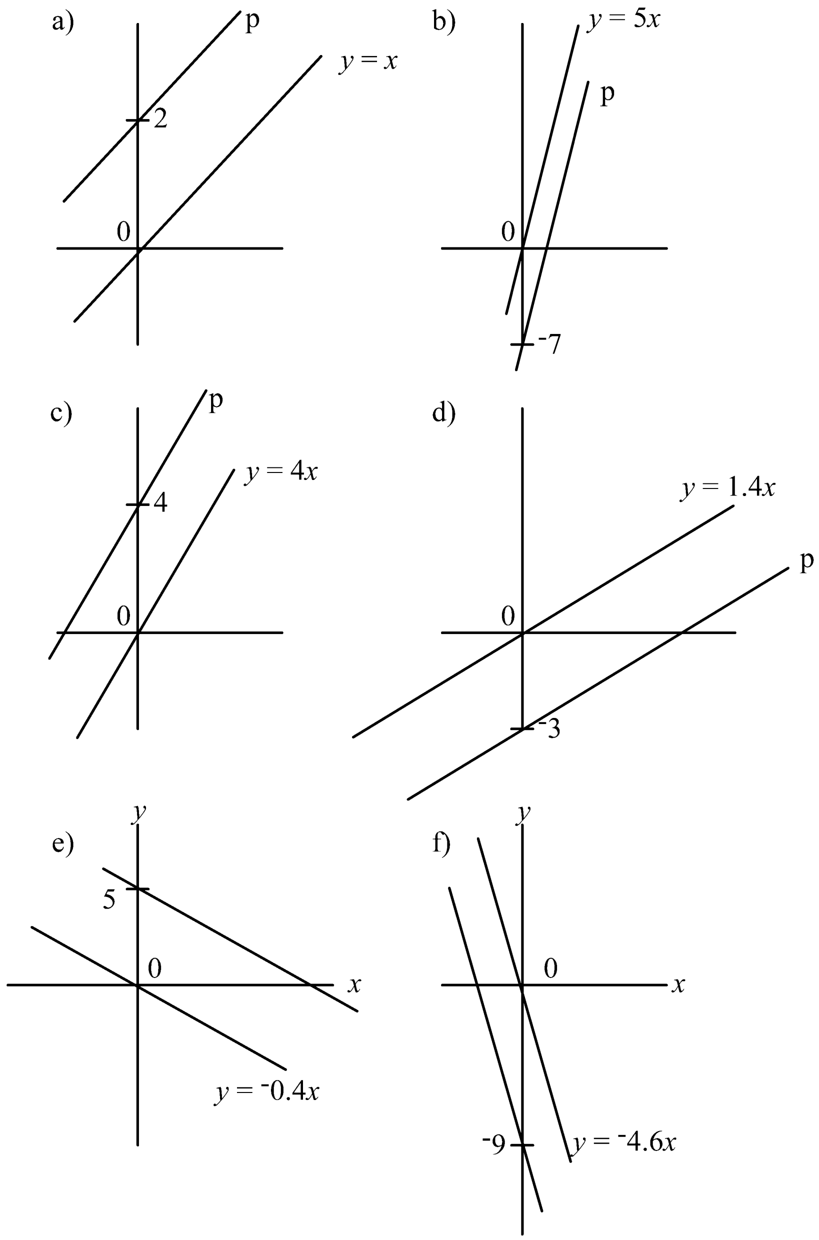
> The y coordinate of the point where the line with equation $y = ax + b$ crosses the y-axis is called the ***y*-intercept** of the line.

2. What is the y-intercept of the lines with equation

   a) $y = 7x - 9$ b) $y = 7 + 9x$ c) $y = x + 16$ d) $y = {}^{-}8 + 3x$

   e) $y = {}^{-}3x + 5$ f) $y = {}^{-}\frac{1}{3}x - 5\frac{1}{4}$ g) $y = {}^{-}4.3x + 5.6$

3. What is the equation of the line p in each case? Line p is parallel to the line through O.



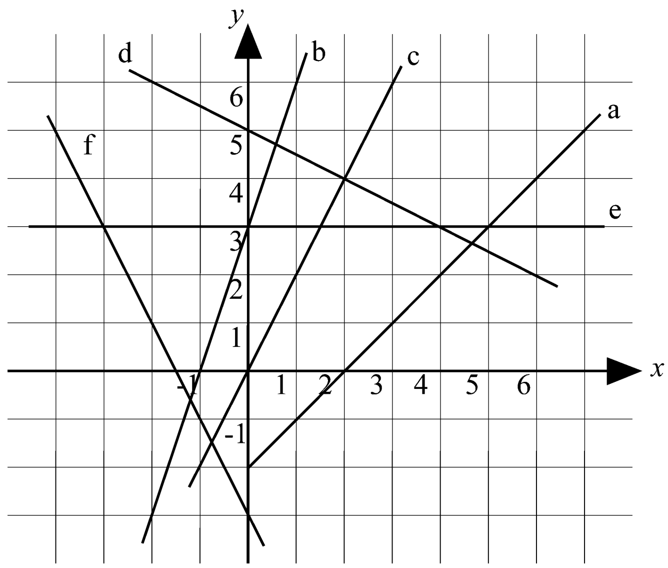
In the equation of the line $y = ax + b$:

a is the gradient or step size and b is the y-intercept.

The format of an equation of a line in word formula:

y = (gradient) x + (y-intercept)

4. For each line a, b, c, d, e and f
   a) What is the y-intercept of the line?
   b) What is the gradient of the line?
   c) What is the equation of the line?



Section B2: Plotting straight line graphs, given the equation of the line

In the plotting of straight line graphs, given their equation, the y-intercept and x-intercept are frequently included as they can be computed fairly easily (putting $x = 0$, or $y = 0$ respectively).

Pupils should be given sufficient practice in plotting graphs.

To plot the graph of the line with equation $y = ax + b$, you need the coordinates of three points (in theory, two are sufficient, but students make fewer errors if they choose three points).

Choose the following three points (if possible):

(i) the point where the line crosses the y-axis

(ii) the point where the line crosses x-axis

(iii) any other point

The x coordinate of the point where the line crosses the x-axis is called the x intercept of the line.

For example, to plot the graph of the line with equation $y = 3x - 6$:
$x = 0 \Rightarrow y = {}^{-}6$ (the y intercept)
$y = 0 \Rightarrow 0 = 3x - 6 \Rightarrow x = 2$ (the x intercept)
$x = 4 \Rightarrow y = 3 \times 4 - 6 = 12 - 6 = 6$.

Tabulated:

x	0	2	4
y	$^{-}6$	0	6

Section B3: Drawing lines with equations *px* + *qy* = *r*

The relationship between the y coordinate and x coordinate can be given in a form different from $y = ax + b$.

To draw the graph of lines with equations written in a different format, for example as

$x + y = 4$ or $2x + y = 7$ you follow the normal four steps:

(i) find and tabulate the coordinates of three points on the line: the point where the line crosses the y-axis, where it crosses the x-axis and one other point

(ii) plot the points on a labelled coordinate grid. Use x (two small crossing lines) to indicate the position of the point

(iii) join the points, extending the line at both sides

(iv) label the line with its equation

Self mark exercise 4

1. Answer the questions in worksheet 7.
2. Using the same coordinate grid draw and label the lines with the following equations. Follow the steps outlined above.

   a) $y = 2x - 5$ b) $y = 4 - 4x$ c) $y = x + 2$ d) $y = \frac{-1}{2}x + 3$

3. Using the same coordinate grid draw and label the graphs of the lines with these equations:

   a) $x + y = 6$ b) $x + \frac{1}{2}y = {}^{-}4$ c) ${}^{-}2x + 3y = 6$ d) $3x - 4y = 12$

   e) $2x + y = 0$ f) $x - y = {}^{-}5$

Check your answers at the end of this unit

Section B4: Finding equations of lines through two given points

The relationship between the y-coordinate and x-coordinate for non vertical lines can be given in the form $y = ax + b$ or in word formula y = (gradient/step size)x + (y-intercept). A line is fixed if you know two points on the line as there is only one line through two points. To find the equation of the line given two points, find the values of a and b in the relation $y = ax + b$.

Example

Find the equation of the line through the points with coordinates P (1, 2) and Q(3, 10).

Step 1: find the gradient of the line

To move from P to Q: 2 steps to the right and 8 steps up.

If you move 1 step to the right you will move 4 steps up (divided by 2)

The gradient or step size is 4.

Write the equation as you know it now: $y = 4x + b$. (I)

Step 2: find the value of b (the y-intercept)

Substitute the coordinates of either P or Q in (I) and solve the equation for b.

Substituting P(1, 2) gives: $2 = 4 \times 1 + b$

$2 = 4 + b$

Use the cover up method to find b: "What to add to 4 to get 2?"

Answer: $^{-}2$.

So $b = {}^{-}2$

The equation of the line is therefore $y = 4x - 2$. (II)

Check by substituting in (II) the x-coordinates of the other point Q(3, 10) and checking whether the correct y-coordinate is obtained.

If $x = 3$ $y = 4 \times 3 - 2 = 12 - 2 = 10$ which is indeed the y-coordinate of the given point Q.

Self mark exercise 5

1. Find the equation of the lines through the points A and B if

   (i) A(1, 2) and B(3, 6)
   (ii) A(3, 1) and B($^-5$, 7)
   (iii) A($^-2$, 2) and B(0, 10)
   (iv) A($^-5$, 2) and B($^-1$, $^-6$)
   (v) A($^-2$, 2) and B($^-2$, 10)
   (vi) A(3, 1) and B($^-7$, 1)

2. Find the equation of the line through the points A and B if A(x_1, y_1) and B(x_2, y_2).
3. What is (i) the gradient (ii) y-intercept (iii) x-intercept of the line with equation $ax + by = c$, provided $a \neq 0$ and $b \neq 0$ at the same time?
4. Explain what happens if (i) $b = 0$ (ii) $a = 0$ in question 3.

Check your answers at the end of this unit.

Unit 2, Practice activity 3

1. Write a detailed lesson plan and outline for covering with your pupils the finding of equations of straight lines. Ensure you take into account the different attainment levels in your class.
2. Write 5 challenging questions related to finding equations of straight lines.
3. The coordinates of the vertices of triangle ABC are A(0,0), B(6, 3) and C(1, 9).
   a) Find the coordinates of the midpoint D of BC and the equation of the line AD (the median).
   b) Find the coordinates of the midpoint F of AB and the equation of the median CF.
   c) Find the equation of the median BE.
   d) Show that the three medians pass through one point and find the coordinates of this point Z (the centroid of the triangle).
   e) Find the ratio in which the median divide each other, i.e., the ratios AZ :ZD, BZ : ZE and CZ : ZF.
   f) Find the centroid of a triangle with coordinates of the vertices ($x1$, $y1$), ($x2$, $y2$) and ($x3$, $y3$).
4. The coordinates of the vertices of triangle ABC are A(0, 0) B(6, $^-8$) and C(6, 3).
   a) Find the equations of the three altitudes (line from the vertex perpendicular to the opposite side) AD, BE and CF.
   b) Show that the three altitudes are concurrent (pass through one point) and find the coordinates of this point.

Present your assignment to your supervisor or study group for discussion.

Summary

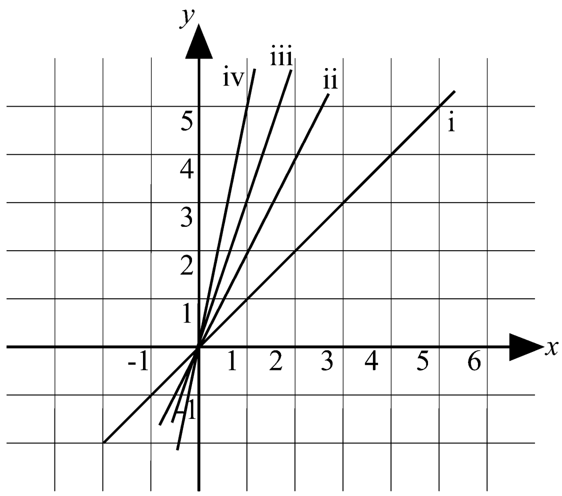
This unit continues the practice of having students learn geometrical properties, such as parallelism, through induction and thorough Cartesian rather than Euclidian concepts. Still, many students at this age experience difficulty making the conceptual links between spatial and algebraic expressions for the same “shape”; the unit recommends lots of exercises!

Unit 2: Answers to self mark exercises

Self mark exercise 1

1. Worksheet 4

   1. a)



   b) $y = 4x$ c) $y = x$ d) coefficient of x

   2. a) $y = 5x$ b) $y = 4x$ c) $y = 3\frac{3}{4}x$ d) n$y = 4.11x$

   3. a) $y = 3x$ b) $y = \frac{1}{2}x$ c) $y = 5x$

   4. a) 1, b) 3. c) 2, d) $\frac{1}{2}$

   5. a) 2, 4, 2

      b) (i) 3 (ii) 2 (iii) 3 (iv) $\frac{4}{3}$ (v) 1 (vi) 0 (vii) undefined

2. No. Vertical lines have the format $x = k$, where k is a constant. This format is not included in the form $y = ax + b$ which is a general equation for all line NOT parallel to the y-axis.
3. Horizontal, $y = p$, where p is a constant.
4. Vertical lines, $y = k$, where k is a constant
5. a) 0, $y = 0$ b) undefined, $x = 0$
6. See 7a

7. a) (i) $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-(y_2 - y_1)}{-(x_2 - x_1)} = \dfrac{y_1 - y_2}{x_1 - x_2}$

   (ii) division by zero is undefined

   b) tangent c) $\tan\theta = \dfrac{x}{y}$ = gradient

   d) if the angle increases from 0° to 90° the gradient increases from 0 to infinity.

   if the angle increases from 90° to 180° the gradient increases from negative infinity to 0

Self mark exercise 2

1. Worksheet 5

   1. gradient 1, the lines are parallel
   2. gradient 2, the lines are parallel
   3. gradient $\frac{1}{2}$, the lines are parallel
   4. gradient $^{-}2$, the lines are parallel
   5. gradient 0, the lines are parallel
   6. undefined gradient (or gradient ∞)
   7. (a) $y = 4x - 7$ and $y = 4x$ (b) $y = 2x + 3$ and $y = 4 + 2x$

      (c) none (d) $y = 2$ and $y = {}^{-}2$ (e) 4, 2, no parallel lines, 0
   8. (a) $\frac{1}{3}$ (b) 3 (c) $-\frac{1}{2}$ (d) 0

   If two lines are parallel lines then they have equal gradients.

2. If lines have equal gradient they are parallel.
3. If lines are parallel they have equal gradients.
4. inductive: conjectures were made from a number of specific examples

Self mark exercise 3

1. Worksheet 6

   1. a) $^{-}3$, $\frac{1}{3}$. Product $^{-}1$ b) 4, $-\frac{1}{4}$. Product $^{-}1$

      c) $-\frac{1}{2}$, 2. Product $^{-}1$ d) undefined, 0. Product undefined

      e) lines are parallel to the coordinate axes/horizontal, vertical line

      f) the product of their gradients is $^{-}1$, provided lines are not horizontal/ vertical

   2. a) 0, undefined; product undefined

      b) 90°, lines are perpendicular but product of their gradient is not $^{-}1$

   3. a) $^{-}1$, 1; product $^{-}1$

b) 90°, product of gradients is $^{-}1 \Rightarrow$ the lines are perpendicular to each other

4. a) $^{-}2$, $\frac{1}{2}$; product gradients is $^{-}1$

   b) 90°, product of gradients is $^{-}1 \Rightarrow$ the lines are perpendicular to each other

5. a) $^{-}3$, $\frac{1}{3}$; product gradients is $^{-}1$

   b) 90°, product of gradients is $^{-}1 \Rightarrow$ the lines are perpendicular to each other

6. a) $y = {}^{-}\frac{1}{4}x - 1$ and $y = 4x - 7$ b) $y = 2x + \frac{1}{2}$ and $y = 2 - \frac{1}{2}x$

   c) $y = 3 - x$ and $y = {}^{-}3 + x$ d) $y = {}^{-}2$ and $x = {}^{-}4$; $y = \frac{1}{2}$ and $x = {}^{-}4$

2. If two lines are perpendicular to each other (but not horizontal/vertical) then the product of their gradients if $^{-}1$.

3. If the product of the gradients of two lines is $^{-}1$ then the two lines are perpendicular to each other.

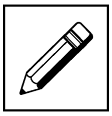
4. a) See text, gradient p is $\frac{b}{a}$, gradient q is $^{-}\frac{a}{b}$; vertical lines have no gradients, yet they are perpendicular to lines that are horizontal (gradient 0). The product is however not $^{-}1$.

   b) A(1, m) B(1, $^{-}\frac{1}{m}$)

   $|OA|^2 = 1 + m^2$ $|OB|^2 = 1 + \frac{1}{m^2}$ $|AB|^2 = (m + \frac{1}{m})^2$

   $= m^2 + 2 + \frac{1}{m^2}$

   Hence $|OA|^2 + |OB|^2 = |AB|^2$



Self mark exercise 4

1. Worksheet 7

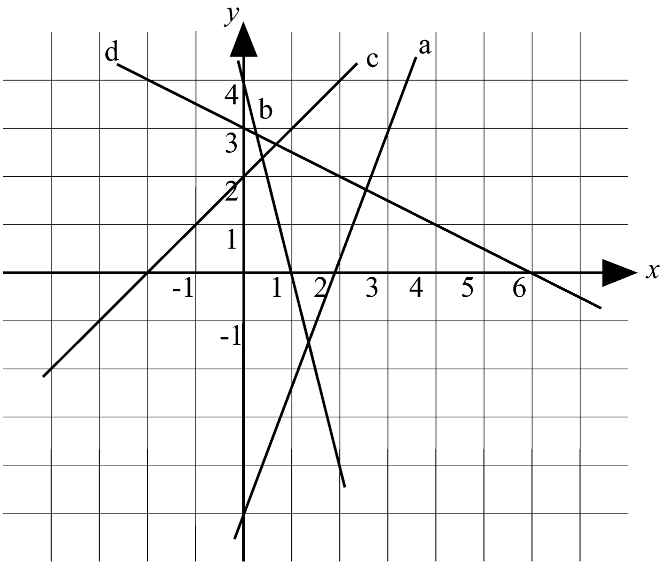
1. b) $^{-}4$, $^{-}2$, 0, 3
   c) 9 d) $^{-}56$ e) p

2. a) $^{-}9$ b) 7 c) 16
   d) $^{-}8$ e) 5 f) $^{-}5\frac{1}{4}$
   g) 5.6

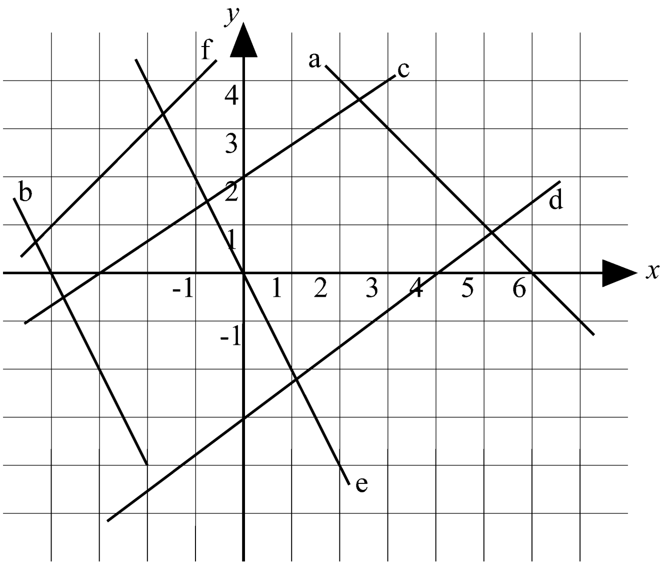
3. a) $y = x + 2$ b) $y = 5x - 7$ c) $y = 4x + 4$
   d) $y = 1.4x - 3$ e) $y = 0.4x + 5$ f) $y = {}^{-}4.6x - 9$

4. a) $^-2, 1, y = x - 2$ b) $2, 3, y = 3x + 2$ c) $0, 2, y = 2x$
d) $1, ^-2, y = x - 2$ e) $0, 3, y = 3$ f) $^-2, ^-3, y = 2x - 3$

2.



3.



Self mark exercise 5

1. (i) $y = 2x$ (ii) $y = {}^-\frac{3}{4}x + 3\frac{1}{4}$ (iii) $y = 4x + 10$

(iv) $y = {}^-2x - 8$ (v) $x = {}^-2$ (vi) $y = 1$

2. Step 1: gradient = $\dfrac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}} = \dfrac{y_1 - y_2}{x_1 - x_2}$

$y = \dfrac{y_1 - y_2}{x_1 - x_2}\, x + b$ (i)

Step 2: Substitute (x_1, y_1) to find b. $y_1 = \frac{y_1 - y_2}{x_1 - x_2} x_1 + b$

$b = y_1 - \frac{y_1 - y_2}{x_1 - x_2} x_1$

Placing the value of b in equation (i)

$y = \frac{y_1 - y_2}{x_1 - x_2} x + y_1 - \frac{y_1 - y_2}{x_1 - x_2} x_1$

$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2} (x - x_1)$

3. (i) $-\frac{a}{b}$ (ii) $\frac{c}{b}$ (iii) $\frac{c}{a}$

4. If $b = 0$, the equation represents a vertical line. These lines have undefined gradient and no y-intercept.

   If $a = 0$, the equation represents a horizontal line. These lines have gradient 0 and no x-intercept.

Unit 3: Transformations I

Purpose of Unit 3

The purpose of this unit is to revise your knowledge on transformations: reflection, rotation, translation and enlargement. In the classroom situation reflection and rotation require a practical approach: folding and rotating tracing paper. A practical and intuitive approach of the linear transformations is used in this unit. The transformations can be done on plain or on coordinate grid relating the topic to coordinate geometry. The emphasis is on how you could present the topic to pupils in the class and the misconceptions of pupils to be aware of. Going through this unit might also help you to get your own understanding of reflection, rotation, translation and enlargement more clear.

Objectives

When you have completed this unit you should be able to:

- set activities at different levels of difficulty to pupils to draw reflections of given shapes in given mirror lines
- set activities at different levels of difficulty to pupils to draw the line of reflection given a shape and its image under a reflection
- find the equation of the mirror line given an object and its image on a coordinate grid
- list four properties of reflection, i.e.,

  In a reflection

  (i) the line segment connecting a point with its image is perpendicular to the mirror line (unless the point is on the mirror line)

  (ii) a point and its image are at the same distance from the mirror line (or the mirror line bisects the line segment joining a point and its image)

  (iii) the image of a line (segment) parallel to the mirror line is parallel to the mirror line

  (iv) the image m′ of a line m (not parallel to the mirror line) meets m on the mirror line
- develop a discovery activity for pupils to discover the above four properties of reflection
- state the common errors/misconceptions of pupils when reflecting a shape
- take appropriate remedial steps to overcome pupils' misconceptions in reflection of shapes
- set activities at different levels of difficulty for pupils to draw the image of a shape under rotation (multiples of 90°) given the shape and its centre

- describe fully a rotation given the shape and its image under a rotation
- state common difficulties of pupils in rotating of shapes
- take appropriate remedial steps to overcome pupils' misconcepts in the rotation of shapes
- use column vectors to describe translations
- translate shapes given the shape and the translation vector
- state the different notations used to denote the translation vector
- use games to consolidate the concept of translation
- set a variety of activities at different levels for pupils to develop understanding of enlargements
- distinguish between shapes that are similar and shapes that are enlargements from each other

Time

To study this unit will take you about 10 hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you have planned to cover the topic.

Section A: Reflection

Section A1: Reflection on reflection

Before you start on this unit reflect on your present practice in your classroom when covering the topic of ‘reflection’. The following questions might guide you.

1. Write down an outline of the lessons in which the concept ‘reflection of shapes’ is covered. What is your starting point? What are the activities the pupils are involved in?
2. What do you cover and how do you do it? Do you cover for example: Reflections on square grid paper ? Reflections on plain paper? Reflections in horizontal, vertical, slanting lines? Reflections of shapes without reference to coordinates? Reflections of shapes and points with reference to the coordinates?
3. Write down all the properties of a reflection. Which of the properties do you want pupils to learn?
4. Write down the common errors made by pupils when reflecting shapes or finding lines of reflection. What do you do to prevent these common errors?

When going through this unit keep on returning to what you wrote down on your own practice.

Section A2: How to draw a reflection of a shape

What method do you use when having to obtain the reflection of a shape in a given mirror line? What method do you present to your pupils? Work the following exercise and reflect on HOW you are obtaining the image of each shape.

Self mark exercise 1

1. Copy the following shapes and find accurately their reflection in the mirror line (or line of reflection) m

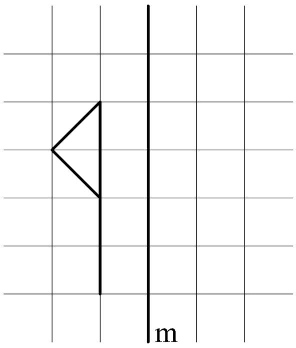


Figure 1

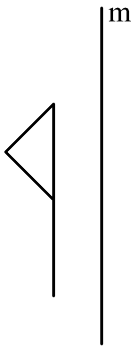


Figure 2

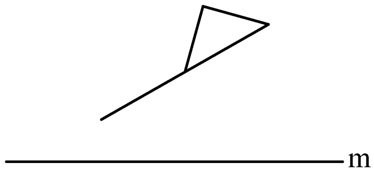


Figure 3

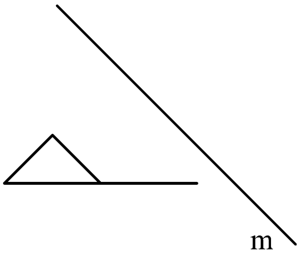


Figure 4

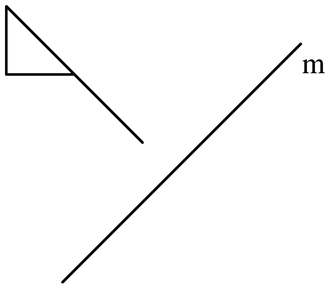


Figure 5

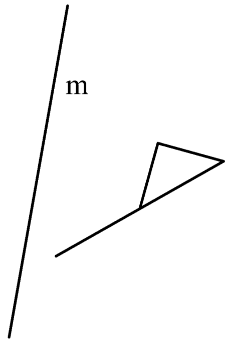
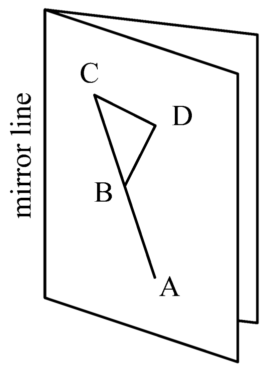


Figure 6

Check your answers at the end of this unit.

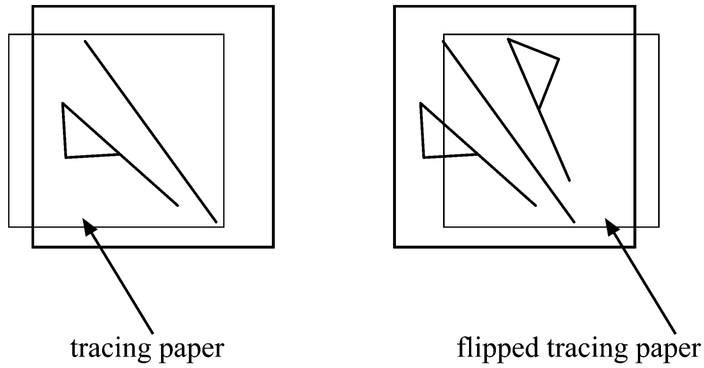
In the **initial stage** pupils are expected to 'intuitively' find the image of the object in the mirror line by

(i) folding the paper along the mirror line and using the point of a compass to mark the 'corners' of the reflected shape.



Piercing at A, B, C and D will give A′, B′, C′ and D′ at the other side of the mirror line. Joining these points will give the mirror image of the object.

(ii) using tracing paper. Place the tracing paper on top of the original and copy mirror line and object. Now flip the tracing paper and ensure the mirror line on the tracing is on top of the original mirror line. Piercing with compasses or sharp pencil will mark the position of the image.

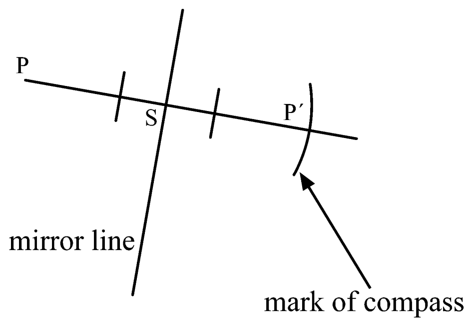


In a **later stage**, after pupils have discovered some of the properties of reflection, set square and ruler can be used.

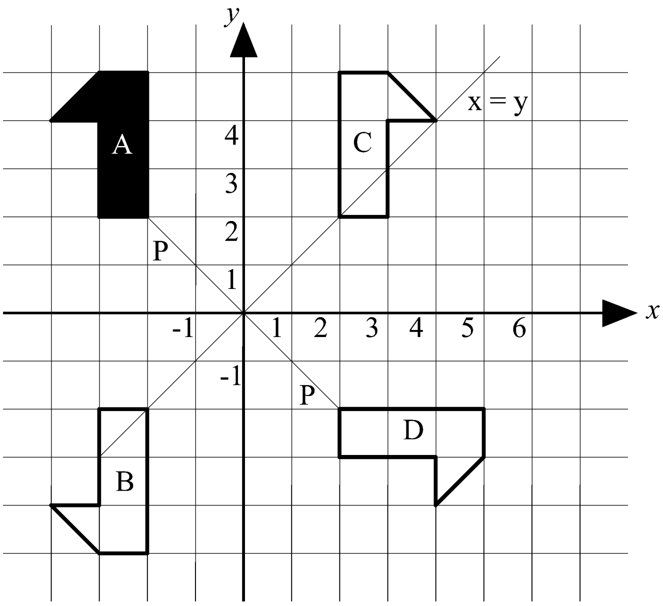
In turn the image of 'corner' points of the shape are found by

(i) drawing a line from the point P perpendicular to the mirror line (meeting the mirror line at S).

(ii) using a compass to ensure that PS = P′S, by placing point at S, making the opening equal to PS and circling at the other side to find P′. The process is repeated for all 'corner' points of the object.

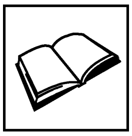


The following diagram is to remind you of reflection on a square grid.



The diagram illustrates reflection of shape A, the **object**, onto B, the **image**, by reflecting in the x-axis (the line with equation $y = 0$).
C is the image of A under a reflection in the y-axis (the line with equation $x = 0$).
D is the image of A under a reflection in the line with equation $x = y$.

To describe a reflection fully you are to give the equation of the **line of reflection** or **mirror line**.



Section A3: Pupils' difficulties with reflections

Difficulty level of questions on reflection of shapes in a given mirror line depends on

(i) ***Presence or absence of a grid.***
With grid lines given the facility increases.

(ii) ***The complexity of the shape to be reflected.***
When the shape consist of a single point the question of reflecting the point in a given mirror line is easier than when a more complicated shape is to reflected.

(iii) ***Slope of line of reflection/mirror line.***
Whether or not the mirror lines are horizontal/vertical or skew has an impact on the difficulty of the questions. In the latter case the questions become harder.

See the following illustration. Figure 1 is easier than the item in Figure 2, which is still easier than Figures 3 and 4.

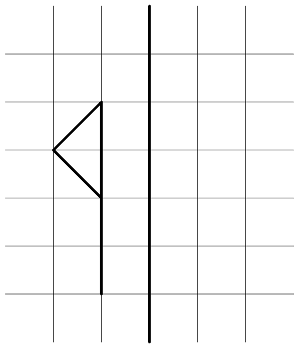


Figure 1

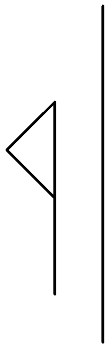


Figure 2

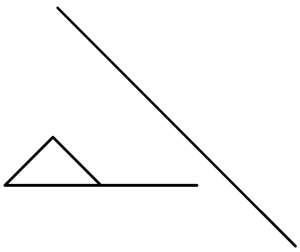


Figure 3

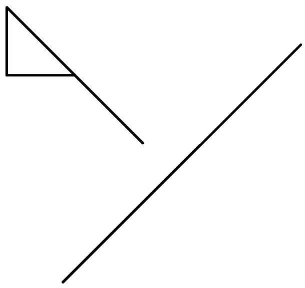


Figure 4

The same is true in cases where the object and the image is given, and the pupil is required to draw the mirror line, if any. (see Fig 5, 6, and 7).

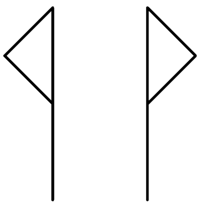


Figure 5

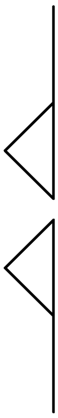


Figure 6

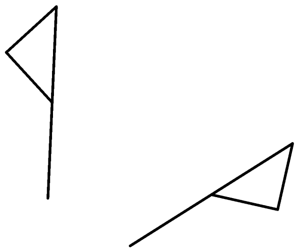


Figure 7

(iv) **Mirror line passes through the shape to be reflected or contains one of the line segments of the shape.**

Questions in which the mirror line has no common points with the shape to be reflected are easier for pupils to handle.

In the following figure, the situation in Figure 1 is easier than the situation in Figure 2.

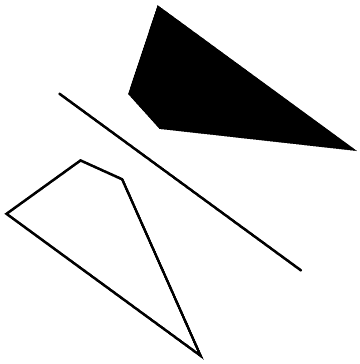


Figure 1

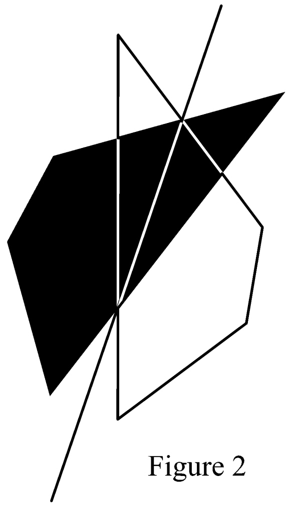
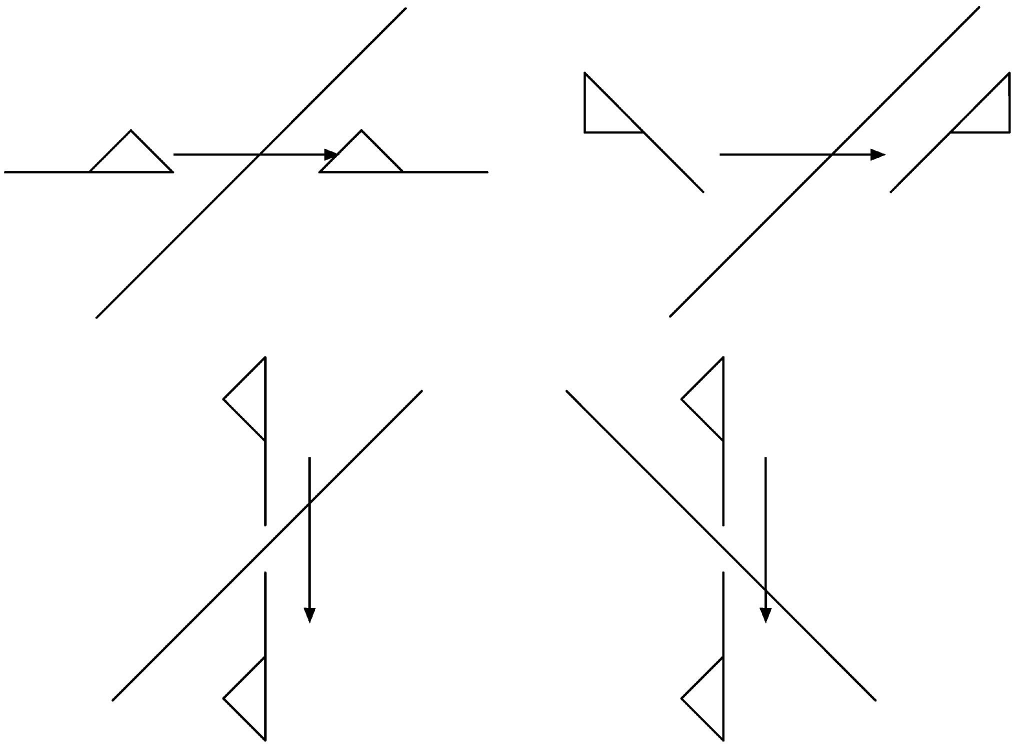


Figure 2

Section A4: Common errors of pupils

a) When the given mirror line is slanting the slope is ignored and the object is reflected horizontally or vertically. The presence or absence of a grid has no influence on this error.

Error workings are illustrated below.



b) If the object is horizontal or vertical the tendency to reflect horizontally (vertically) in a slanting mirror line is even stronger. In general there is a tendency to draw images parallel to the object.

Unit 3, Practice activity 1

1. Design a diagnostic test to find out whether pupils in your class have any of the misconceptions described on the previous page. Ensure all possible cases are covered: reflection with or without grid, line of reflection horizontal, vertical, slanting, position of the object with respect to the line of reflection. For each item state the objective: What error is the item to diagnose?
2. Administer the test and analyse the results.
3. A pupil makes the error described and illustrated above: the object is reflected horizontally or vertically with a line of reflection slanting.

   Describe in detail the four steps in remediation ((i) diagnosing the error by asking the pupil to explain what he/she did (ii) creating conflict in the mind of the pupil by using the pupil's method which clearly leads to a mental conflict situation (iii) setting activities to build the correct concepts and (iv) consolidation of the concept) you would take to help the pupil to overcome the error.

Present your assignment to your supervisor or study group for discussion.

Section A5: Objectives to be covered with pupils

Pupils should be able to

a) given the mirror line (line of reflection) and the shape, to find the image of the shape when reflected in the line (drawing or giving coordinates)

b) given the shape, the line of reflection and an 'image', to determine whether the given line is/is not a line of reflection

c) given a shape and an image of the shape, to find the mirror line (if any), i.e., draw the line and/or find its equation

Each objective is to be covered without a grid present and on a coordinate grid, with the three possible positions of the line of reflection: horizontal, vertical and slanting.

Properties of reflections that could be covered are:

(i) the line segment connecting a point with its image is perpendicular to the mirror line (unless the point is on the mirror line)

(ii) a point and its image are at the same distance from the mirror line (or the mirror line bisects the line segment joining a point and its image)

(iii) the image of a line (segment) parallel to the mirror line is parallel to the mirror line

(iv) the image m´ of a line m (not parallel to the mirror line) meets m on the mirror line

An investigative approach is to be used in covering these properties.

Section A6: Worksheets for pupils on reflection

On the next pages you will find suggestions for activities that can be presented to the pupils.

Worksheet 1 has as objectives (i) to practice/consolidate drawing of images of shapes given the shape and the mirror line (ii) to discover the following properties of a reflection:

a) the line segment connecting a point with its image is perpendicular to the mirror line

b) a point and its image are at the same distance from the mirror line (or the mirror line bisects the line segment joining a point and its image)

c) a point on the mirror is its own image

Worksheet 2 has as objectives (i) to practice/consolidate drawing of images of shapes given the shape and the mirror line (ii) to discover that a) the image m′ of a line m (not parallel to the mirror line) meets it on the mirror line b) the image of a line (segment) parallel to the mirror line is parallel to the mirror line.

Worksheet 3 has as objectives (i) to draw the line of reflection given a shape and its image (ii) to recognise whether or not a diagram represents a reflection.

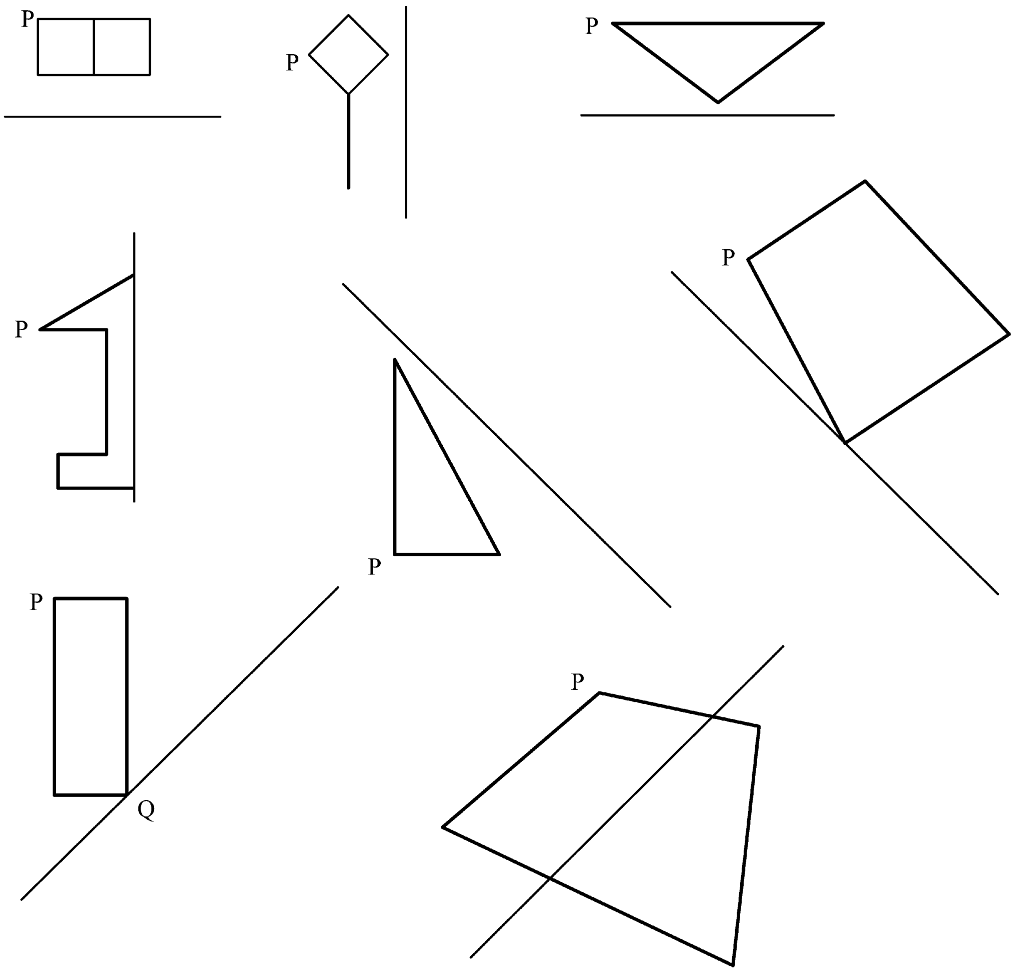
Worksheet 4 looks at reflection of points given the coordinates on a coordinate grid. Pupils are the discover relationships between the coordinates of the original point P(a, b) and the image of P when P is reflected in (i) the x-axis (ii) the y-axis (iii) the line with equation $x = y$ (iv) the line with equation $y = {}^{-}x$.

Worksheet 1

1. Copy the diagrams on a A4 paper, using tracing paper to trace the diagrams including the mirror line.

   Flip over your tracing paper and place mirror line of tracing on the mirror line of the original.

   Mark the positions of the image ‘corners’ and join them to show the position of the shape after it has been reflected. Mark the image of P, P′.

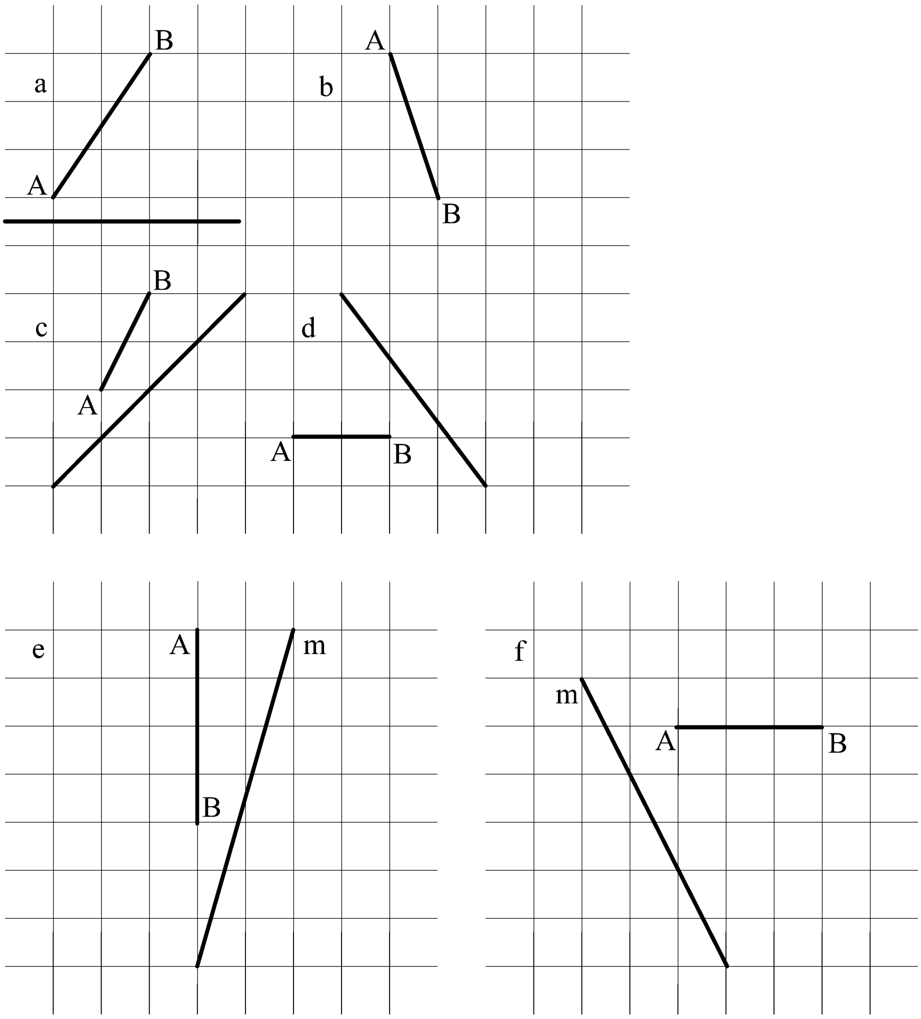


2. Draw in each diagram the line segment PP′.
3. In each case: What is the angle between the segment PP′ and the line of reflection?
   Complete the statement:
   In a reflection the line joining a point P with its P′ is ......
4. Mark in each diagram the point where the line PP′ meets the line of reflection S.
   Measure and compare the lengths of PS and P′S. What do you notice?

   Complete the statement:
   In a reflection the distance from P to the mirror line is ......
5. What is the image of the point Q? Make a statement about the image of points on the mirror line.

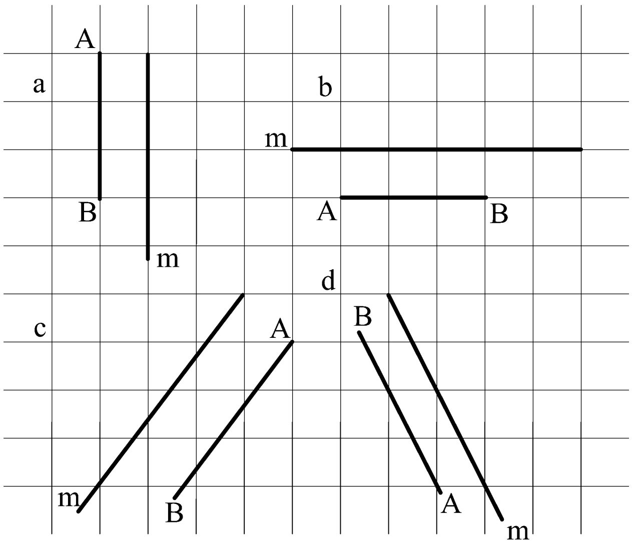
Worksheet 2

1. Copy the following diagrams on squared paper and draw the reflection of the line segments AB in the given mirror line. Label the image A´B´. Produce the line segment and its image until they meet.



Compare the results in your group and complete the statement:
In a reflection a line and its image meet ......

2. Check your statement for the following cases.
   Copy the diagram on squared paper and reflect AB. Find where AB and its image A´B´ meet.



   Where do AB and A´B´ meet?
   Complete the statement:
   If a line is parallel to the mirror line the image of the line is ......

Worksheet 3

1. In each of the following diagrams one shape is the reflection of the other. Copy on squared paper and draw the position of the mirror line.