Module 3

# Junior Secondary Mathematics

## Shapes and Sizes

The Commonwealth of Learning

**Science, Technology and Mathematics Modules**

**for Upper Primary and Junior Secondary School Teachers of Science, Technology and Mathematics by Distance in the Southern African Development Community (SADC)**

**Developed by**
**The Southern African Development Community (SADC)**

**Ministries of Education in:**

- **Botswana**
- **Malawi**
- **Mozambique**
- **Namibia**
- **South Africa**
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Published jointly by The Commonwealth of Learning and the SADC Ministries of Education.

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ISBN 1-895369-65-7

## SCIENCE, TECHNOLOGY AND MATHEMATICS MODULES

This module is one of a series prepared under the auspices of the participating Southern African Development Community (SADC) and The Commonwealth of Learning as part of the Training of Upper Primary and Junior Secondary Science, Technology and Mathematics Teachers in Africa by Distance. These modules enable teachers to enhance their professional skills through distance and open learning. Many individuals and groups have been involved in writing and producing these modules. We trust that they will benefit not only the teachers who use them, but also, ultimately, their students and the communities and nations in which they live.

The twenty-eight Science, Technology and Mathematics modules are as follows:

**Upper Primary Science**
Module 1: *My Built Environment*
Module 2: *Materials in my Environment*
Module 3: *My Health*
Module 4: *My Natural Environment*

**Junior Secondary Science**
Module 1: *Energy and Energy Transfer*
Module 2: *Energy Use in Electronic Communication*
Module 3: *Living Organisms' Environment and Resources*
Module 4: *Scientific Processes*

**Upper Primary Technology**
Module 1: *Teaching Technology in the Primary School*
Module 2: *Making Things Move*
Module 3: *Structures*
Module 4: *Materials*
Module 5: *Processing*

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Module 2: *Systems and Controls*
Module 3: *Tools and Materials*
Module 4: *Structures*

**Upper Primary Mathematics**
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Module 2: *Fractions*
Module 3: *Measures*
Module 4: *Social Arithmetic*
Module 5: *Geometry*

**Junior Secondary Mathematics**
Module 1: *Number Systems*
Module 2: *Number Operations*
Module 3: *Shapes and Sizes*
Module 4: *Algebraic Processes*
Module 5: *Solving Equations*
Module 6: *Data Handling*

## A MESSAGE FROM THE COMMONWEALTH OF LEARNING

The Commonwealth of Learning is grateful for the generous contribution of the participating Ministries of Education. The Permanent Secretaries for Education played an important role in facilitating the implementation of the 1998-2000 project work plan by releasing officers to take part in workshops and meetings and by funding some aspects of in-country and regional workshops. The Commonwealth of Learning is also grateful for the support that it received from the British Council (Botswana and Zambia offices), the Open University (UK), Northern College (Scotland), CfBT Education Services (UK), the Commonwealth Secretariat (London), the South Africa College for Teacher Education (South Africa), the Netherlands Government (Zimbabwe office), the British Department for International Development (DFID) (Zimbabwe office) and Grant MacEwan College (Canada).

The Commonwealth of Learning would like to acknowledge the excellent technical advice and management of the project provided by the strategic contact persons, the broad curriculum team leaders, the writing team leaders, the workshop development team leaders and the regional monitoring team members. The materials development would not have been possible without the commitment and dedication of all the course writers, the in-country reviewers and the secretaries who provided the support services for the in-country and regional workshops.

Finally, The Commonwealth of Learning is grateful for the instructional design and review carried out by teams and individual consultants as follows:

- Grant MacEwan College (Alberta, Canada):
  General Education Courses
- Open Learning Agency (British Columbia, Canada):
  Science, Technology and Mathematics
- Technology for Allcc. (Durban, South Africa):
  Upper Primary Technology
- Hands-on Management Services (British Columbia, Canada):
  Junior Secondary Technology

*Dato' Professor Gajaraj Dhanarajan*
President and Chief Executive Officer

## ACKNOWLEDGEMENTS

The Mathematics Modules for Upper Primary and Junior Secondary Teachers in the Southern Africa Development Community (SADC) were written and reviewed by teams from the participating SADC Ministries of Education with the assistance of The Commonwealth of Learning.

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## Introduction

Welcome to *Shapes and Sizes,* Module 3 of Teaching Junior Secondary Mathematics! This series of six modules is designed to help you to strengthen your knowledge of mathematics topics and to acquire more instructional strategies for teaching mathematics in the classroom.

The guiding principles of these modules are to help make the connection between theoretical maths and the use of the maths; to apply instructional theory to practice in the classroom situation; and to support you, as you in turn help your students to apply mathematics theory to practical classroom work.

## Programme Goals

This programme is designed to help you to:

- strengthen your understanding of mathematics topics
- expand the range of instructional strategies that you can use in the mathematics classroom

## Programme Objectives

By the time you have completed this programme, you should be able to:

- develop and present lessons on the nature of the mathematics process, with an emphasis on where each type of mathematics is used outside of the classroom
- guide students as they work in teams on practical projects in mathematics, and help them to work effectively as a member of a group
- use questioning and explanation strategies to help students learn new concepts and to support students in their problem solving activities
- guide students in the use of investigative strategies on particular projects, and thus to show them how mathematical tools are used
- guide students as they prepare their portfolios about their project activities

## How to work on this programme

As is indicated in the programme goals and objectives, the programme provides for you to participate actively in each module by applying instructional strategies when exploring mathematics with your students and by reflecting on that experience. In other words, you "put on your student uniform" for the time you work on this course.

### *Working as a student*

If you completed Module 1…did you in fact complete it? That is, did you actually do the various Assignments by yourself or with your students? Did you write down your answers, then compare them with the answers at the back of the module?

It is possible to simply read these modules and gain some insight from doing so. But you gain far more, and your teaching practice has a much better chance of improving, if you consider these modules as a *course of study* like the courses you studied in school. That means engaging in the material—solving the sample problems, preparing lesson plans when asked to and trying them with your students, and so on.

To be a better teacher, first be a better student!

### *Working on your own*

You may be the only teacher of mathematics topics in your school, or you may choose to work on your own so you can accommodate this programme within your schedule. Module 1 included some strategies for that situation, such as:

1. Establish a regular schedule for working on the module.

2. Choose a study space where you can work quietly without interruption.

3. Identify someone whose interests are relevant to mathematics (for example, a science teacher in your school) with whom you can discuss the module and some of your ideas about teaching mathematics. Even the most independent learner benefits from good dialogue with others: it helps us to formulate our ideas—or as one learner commented, "How do I know what I'm thinking until I hear what I have to say?"

It is hoped that you have your schedule established, and have also conversed with a colleague about this course on a few occasions already. As you work through Module 3, please continue!

### *Resources available to you*

Although these modules can be completed without referring to additional resource materials, your experience and that of your students can be enriched if you use other resources as well. There is a list of resource materials for each module provided at the end of the module.

## ICONS

Throughout each module, you will find the following icons or symbols that alert you to a change in activity within the module.

Read the following explanations to discover what each icon prompts you to do.

| | | |
|---|---|---|
|  | **Introduction** | Rationale or overview for this part of the course. |
|  | **Learning Objectives** | What you should be able to do after completing this module or unit. |
|  | **Text or Reading Material** | Course content for you to study. |
|  | **Important—Take Note!** | Something to study carefully. |
|  | **Self-Marking Exercise** | An exercise to demonstrate your own grasp of the content. |
|  | **Individual Activity** | An exercise or project for you to try by yourself and demonstrate your own grasp of the content. |
|  | **Classroom Activity** | An exercise or project for you to do with or assign to your students. |
|  | **Reflection** | A question or project for yourself— for deeper understanding of this concept, or of your use of it when teaching. |
|  | **Summary** | |
|  | **Unit or Module Assignment** | Exercise to assess your understanding of all the unit or module topics. |
|  | **Suggested Answers to Activities** | |
|  | **Time** | Suggested hours to allow for completing a unit or any learning task. |
|  | **Glossary** | Definitions of terms used in this module. |

# CONTENTS

# Module 3: Shapes and Sizes

# Module 3
# Shapes and sizes

## Introduction to the module

This module is concerned with shape and size of two dimensional figures. This branch of mathematics is generally referred to as geometry.

Geometry belongs to one of the oldest branches in mathematics. People from the early days of civilisation have been interested in describing physical forms they observed in their environment and looking for similarities in sizes and shapes. Earliest records, on Babylonian clay-tablets, of geometrical activities date back to 3000 BC. The Greeks developed geometry to a great height culminating in the *Elements* of Euclid (*c* 300 BC). This single work had an enormous impact on the future development of geometry. It used an axiomatic deductive method to prove 465 propositions in plane and solid geometry. For many ages hardly anything was added to this great Greek classic.

Expansion of geometry took place again after developments in algebra ($16^{th}$ century in Italy). Desargues (1593 -1661) and Pascal (1623 - 1662) explored a new field in geometry: projective geometry which did not get much attention at the time and was only taken up again in the early nineteenth century. Descartes (1596 -1650) and Fermat (1601 - 1665), at about the same time as Desargues and Fermat developed analytical geometry, a combination of geometry and algebra. Analytical geometry deals algebraically with geometric properties of figures and is a method of geometry. A theorem in geometry is phrased and solved as a corresponding theorem in algebra. The first two units in this module use this method to deal with questions which are basically geometric in nature i.e. dealing with points, lines and closed shapes formed by line segments.

In the nineteenth century a number of geometries, different from the classical Greek Euclidean geometry, had developed. All have as a unifying base the study of properties of shapes that remain unchanged when subjected to a group of transformation. For plane Euclidean geometry the group of transformation are the reflections, rotations, translations and dilatations (enlargements) (NCTM, 1993).

## Aim of the module

The module aims at:

(a) reflection on your present practice in the teaching of basic coordinate geometry and transformation geometry

(b) enhancing your content knowledge so that you may set activities on the geometry topics to your pupils with more confidence

(c) making your teaching of coordinate and transformation geometry more effective by using a pupil centred approach and methods such as group discussion, discovery method, games and investigations.

(d) reflecting on assessment of investigative work

## Structure of the module

This module is divided into four units. Unit 1 and 2 deal with analytical geometry: the method of using algebra in answering geometric questions. In the first unit you will be looking at how the position of an object can be fixed in relation to other objects. In unit 2 you will look at points that are positioned on straight lines and how to describe them. Unit 3 will be looking at movements, in mathematics called transformations, of points and objects. There are many possible movements possible but you will focus on translation, reflection, rotation and enlargement and how to present activities to your students on these topics. Unit 4 will relate the movements studied in unit 3 to algebra. Matrices can be used to describe the movements. The last unit places emphasis on investigative work and hence also looks at how investigative work is to be assessed.

## Objectives of the module

When you have completed this module, you should be able to create a learning environment for your pupils for them to acquire knowledge on:

(i) fixing of the position of objects

(ii) the Cartesian coordinate system as a way to describe position of points

(iii) the Cartesian coordinate system to describe lines and relationships between lines

(iv) transformation geometry in the plane

(v) matrices to describe transformation in the plane

using a pupil centred method in which pupils are actively involved in exploring the Cartesian plane and transformations and appropriate assessment procedure for investigative work.

# Unit 1: Coordinate geometry I

## Introduction

In this unit you will study ways to describe the position of a point. One widely used method is the use of Cartesian coordinates. The use of coordinates is commonly ascribed to René Descartes (1596 - 1650), although it seems that the notation of a coordinate system played little, if any, part in his work. The rectangular coordinates are named after Descartes: Cartesian coordinates. The use of rectangular coordinates to fix location of places is much older and dates back to Hipparchus (*c* 161 - 126 BC). The Cartesian coordinates will be used to fix position of points relative to an origin, calculate distances between points and to obtain coordinates of points positioned in a given way with respect to other given points.

## Purpose of Unit 1

The purpose of this unit is to revise your knowledge on the use of Cartesian coordinates, to extend the concepts and to look at activities that could be used to introduce the concepts to pupils. The main aim is to help you to be a better teacher. When going through this unit (and others) reflect on how the material can be used in your classroom, whether it gives reasons to change your current classroom practice. Assignments are frequently related to trying out material in your classroom

We do hope that you and your pupils will benefit from this unit.

## Objectives

When you have completed this unit you should be able to:

- state and illustrate four different ways to fix the location of a point
- list the concepts to be covered when introducing Cartesian coordinates
- list four points requiring specific attention when using the coordinate grid to plot points
- set activities to your class to introduce Cartesian coordinates
- use games in your class to consolidate Cartesian coordinates
- justify the use of games in the learning and teaching of mathematics
- distinguish between line, line segment and half line or ray
- distinguish between inductive and deductive methods in teaching/learning
- use an inductive or deductive discovery method with your pupils to find the coordinates of the midpoint of a given line segment
- use the Pythagorean theorem in finding distances between points

## Time

To study this unit will take you about 10 hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you have planned to cover the topic.

## Section A: Fixing position

### Section A1: Exploring ways to fix position

1. Write down an outline of the lesson in which you introduce pupils to coordinates. How do you introduce coordinates to pupils, and how do you consolidate the concepts?

   Is the method you use different from the way you were introduced yourself to coordinates as a student? Justify differences and similarities.

2. Use your outline when working through this unit and attach it to your assignment.

The following is an outline of a classroom discussion on fixing position.

**Objective**: to make pupils aware that to fix the position of an object, you require (i) a reference point—you describe the position in relation to another (fixed) point; (ii) distances and/or directions

**Teaching/Learning Aids**: maps (country, Africa, universe), street maps of towns, plan of the school compound, plan of a house are displayed on the wall of the classroom

**Discussion question for pupils**

1. How would they describe to a stranger the location of
   a) their place in the classroom
   b) their house, post office, .... as related to the school
   c) sport field of the school, science block, ... as related to the classroom in which they are sitting
   d) the Head Master's office within the administrative building as related to the entrance of the building
   e) the village/town as positioned in the country
   f) the country as positioned within Africa
   g) the Earth as positioned in the universe (How would you direct aliens to the planet Earth?)
   h) chess pieces on a chess board/draft stones on a draft board

**Follow up questions**

2. a) What do you need to describe the position of something accurately?
   b) How accurate do you have to be?
   c) Can the position of an object be fixed using distances only? If yes: how many distances do you need to know? If No why not?
   d) Can the position of an object be fixed using directions only? If yes, how many directions do you need to know? If no, why not?
   e) How can directions be described?

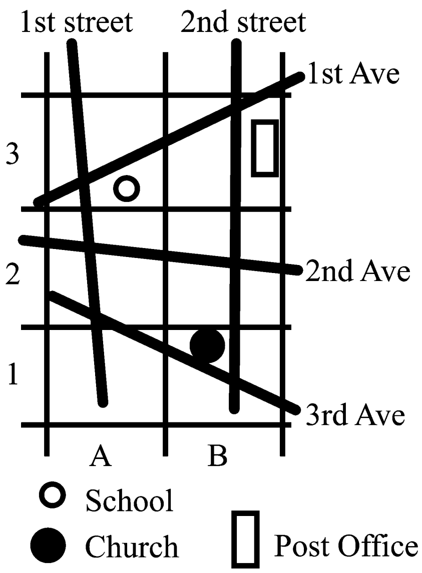
f) Is there a difference between fixing the position of a fly at a certain instance moving in the room and a fly sitting on the ceiling?

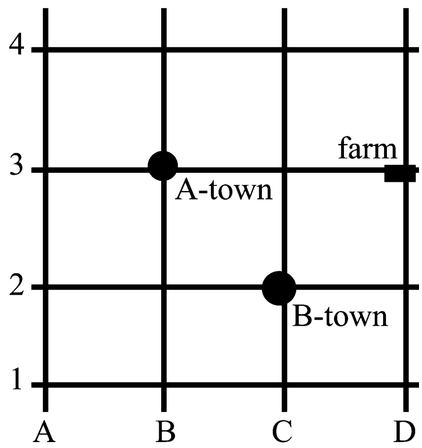
## Self mark exercise 1

1. Answer the above 'follow-up questions' for yourself.
2. On geographical maps, longitude and latitude are used to fix the position of places. Explain the meaning of longitude and latitude. Illustrate with diagrams.

*Check your answers at the end of this unit.*

In the discussion pupils might come up with some of the following methods to fix position. If they did not, you might present the situations to them with the questions as illustrated below. Have the following situations ready as a hand-out for pupils or on an OHP transparency to show to the whole class.




**Questions**

a) Locating by giving the rectangle in which the street/place is found.

   What is the location of the school?
   Answer: in rectangle A3.

   Where can we find $1^{st}$ Street?
   Answer: in rectangles A1, A2 and A3

   Discuss: In which situation is this way of locating accurate enough and suitable?

b) Locating points with two indicators.

   The farm is located at D3

Which town is located at C2?

If C-town is at D4, draw/indicate where C-town is.

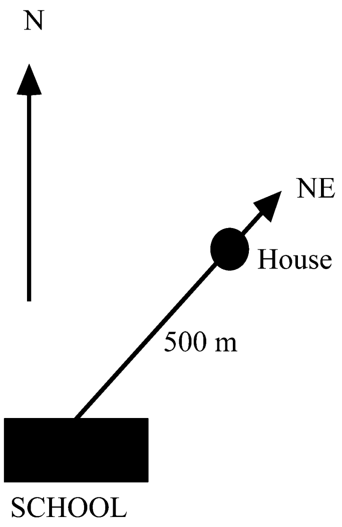
c) Giving the longitude and latitude of a place

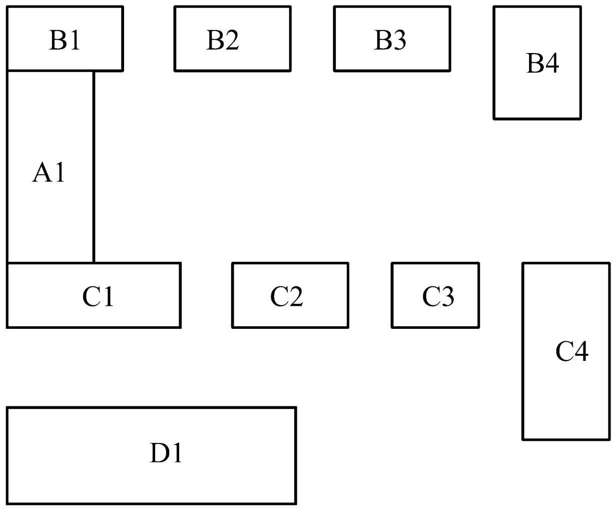
Find on a map the longitude and latitude of the capital of your country.

d) Locating points using distance and direction.

To reach my house from the school you have to walk 500 m from the front of the school in the direction North East (see diagram).

e) Locating by numbering (either single number or letters or combination). In a school compound the buildings/classrooms might be numbered B1, B2 etc.



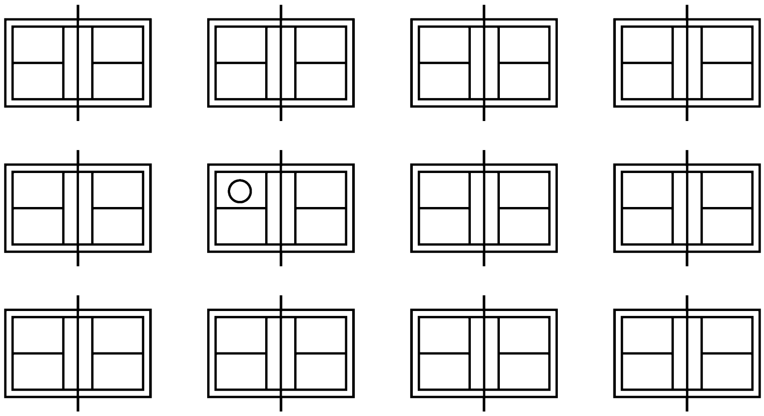

To check pupils understanding the following question can presented:

**P1.** On a sports ground there are 12 fields to play tennis. The layout of the field is presented to pupils on OHP transparency (or hand-out).

Modise is playing on field 8, indicated by o. She has to play her next game on field number 1.

Which field is field 1 according to you? Explain how you numbered the fields.

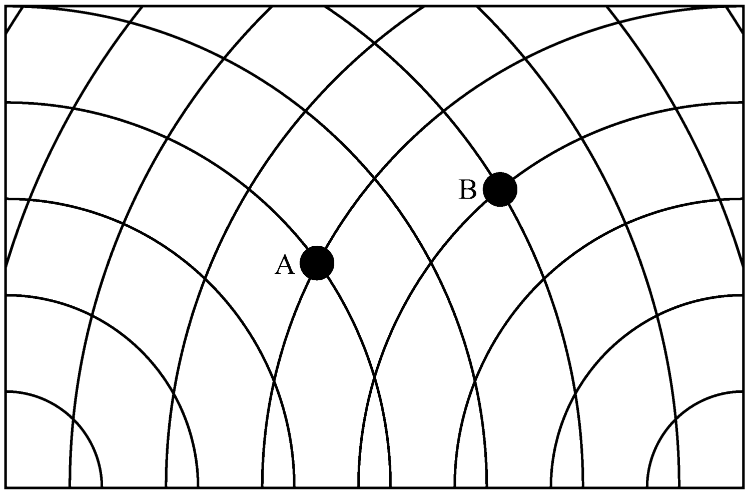
Modise did NOT go to the field you indicated as number 1. She thought field 1 to be another field than the one you indicated. To which field did she go?

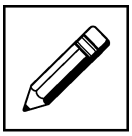


**P2.** Use the following diagram to check pupils' understanding of coordinates in an application.

If A has coordinates (4, 5), what are the coordinates of B?

Plot C (6, 3)



## Self mark exercise 2

1. Answer question **P1** and **P2**.
2. At what level of Bloom's taxonomy (knowledge, comprehension, application, analysis, synthesis or evaluation) do you place each of the questions **P1** and **P2**? Justify.

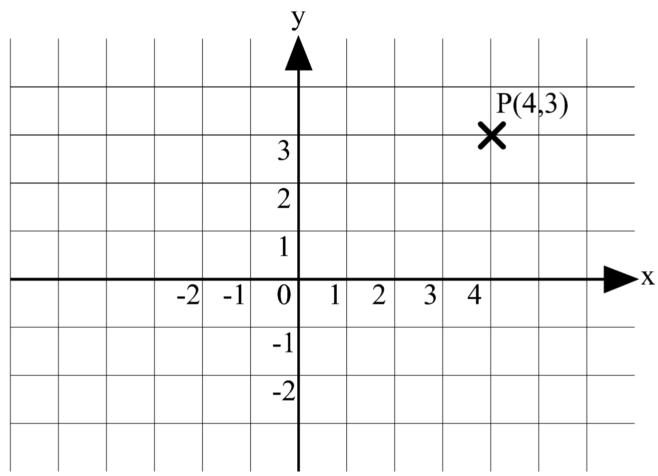
*Check your answers at the end of this unit.*

## Section A2: Cartesian coordinate system

The position of a point in a plane can be fixed by two numbers, called the coordinates of the point. The numbers fix the position relative to a fixed point of origin.

*concepts to be covered :*

> *x*-axis, *y*-axis
> quadrants
> origin
> coordinates (abscissa, ordinate) = (*x*-coordinate, *y*-coordinate)
> scale




Plotting and reading of coordinates is not found difficult by pupils **when the coordinates involved are integers**. It is important to practice plotting points whose coordinates are rational numbers or decimal numbers. One potential problem must be cleared: frequently the axes are referred to as the horizontal and vertical axis. However this only applies to the teacher drawing on the vertical chalkboard! For pupils, working on a horizontal desk top, both axes are horizontal. Some teachers therefore use ‘across’ and ‘up’, before solving the problem by using the *x*-axis and *y*-axis.

Pay also attention to:

(i) axes are to be labelled,

(ii) the origin O is to be indicated,

(iii) along the axis a ‘start’ of the scale is to be indicated

(iv) points are indicated by X (meaning that the point is the intersection of the two short line segments)

(v) the common pupil’s error reading or plotting (*x*, *y*) as (*y*, *x*)

**Consolidation activities to practice** ***plotting of points and reading of Cartesian coordinates:***

**A1** Make various shapes by plotting points on the coordinate grid and joining them.

For example:

On a coordinate grid plot the following points, joining them in the order listed.

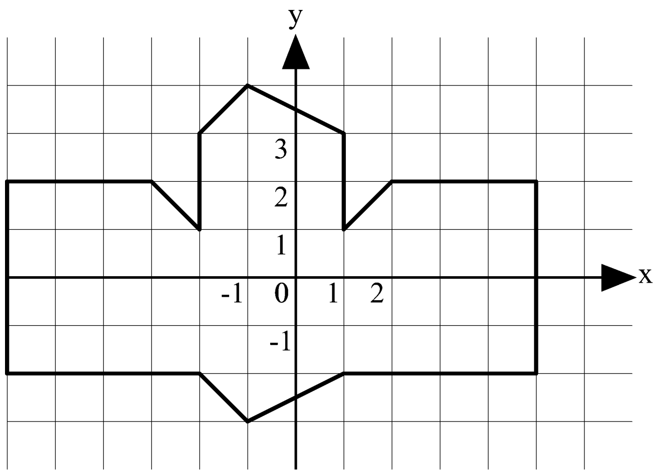
(0, 3), (1, 5), (1, 6), (2, 7), (3, 6), (5, 6), (6, 7), (7, 6), (8, 3), (6, 1), (2, 1), (0, 3)

Now join (3, 2) to (5, 2) and (4, 2) to (4, 3). Finally mark the points (3, 4) and (5, 4).

The activity allows for differentiation. The above example uses positive whole numbers (1st quadrant) only as coordinates. Using fractional, decimal coordinates and coordinates in all the four quadrants can make the question more challenging for higher attainers.

**A2** Read the coordinates in the order required for a given shape to be drawn.

Write down the instructions you would give to your friend to draw the illustrated diagram on a coordinate grid.



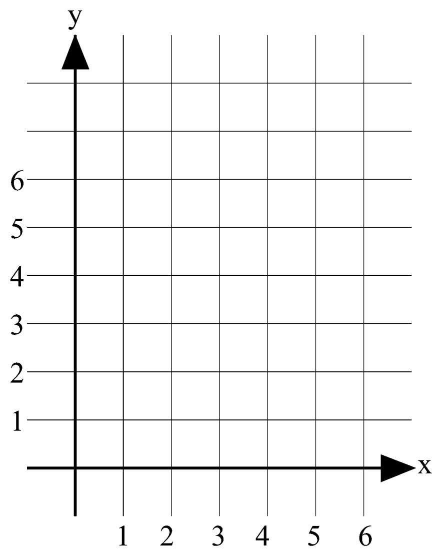

**A3** Pupils make their own shape and write down the sequence of coordinates to be joined to give the shape they designed. Their sequence of coordinates can be given to another pupil to plot. The variety of shapes can make an attractive classroom display, especially when pupils add some colour to their designs.

**A4 Game: Three in a line.**

Objective: Reinforce plotting of points

2 players, coordinate grid, one dice.

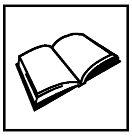
Players take turns in throwing the dice twice, first throw is the $x$-coordinate, second throw is the $y$-coordinate. Players mark points on the grid in different colours. The first player to get three points in a straight line wins the game.




***Variation: Four in a line***

Use a coordinate grid with four quadrants and four dice, two red (negative numbers) and two blue (positive numbers). Players select any two dice from the four and rolls the two chosen dice. The score on the dice determines the coordinates of the point.

The winner is the first player to obtain four points in a line, but not all points are allowed to be in the same quadrant.



**Advantages of using games**

Games can

1. ***Develop a positive attitude towards mathematics.***
   Pupils need to experience: success, excitement, satisfaction, enthusiasm, self-confidence, interest, enjoyment and active involvement. Few media are more successful than games in providing all of these experiences.
2. ***Consolidate mathematical concepts, facts, vocabulary, notation.***
   Concepts, facts, vocabulary, mathematical notation need consolidation. The traditional drill and practice episodes in a lesson are not, in general, very motivating, while a game like environment might consolidate the concepts in an enjoyable and motivating way. In particular games can be used to consolidate mathematical facts, vocabulary and notation.
3. ***Develop mental arithmetic skills.***
   Despite the fact that calculators are a tool in the learning of mathematics, this does not dismiss the need for pupils to know basic number facts and approximate sums, differences, products and quotients. Games can address specifically these important aspects for natural numbers, integers, (decimal) fractions and percent. For example a set of dominoes can be designed to consolidate the equivalence of fractions, conversion of fractions to percent or the four basic operations with integers.

4. ***Develop strategic thinking.***
   Games can encourage pupils to devise winning strategies. Can the person (i) playing first (ii) playing second always win? What is the 'best' move in a given situation?
   For example the game of noughts and crosses. Is there a winning strategy?

5. ***Promote discussion between pupils and between teacher and pupil(s).***
   When certain games are used in the mathematics class as a learning activity, there is a need for discussion: What mathematics did you learn? Is it a good game? Can it be improved?—are some of the questions to look at.

6. ***Encourage co-operation among pupils.***
   Some games require a group playing against another group or a pair of pupils against another pair. Such games can enhance co-operation among the group or the pupils paired. They are to co-operate in order to 'win' the game.

7. ***Contribute to the development of communication skills.***
   In a game the rules need to be explained. Pupils can explain the rules to others orally, formulate rules in writing, describe strategies used to each other—activities enhancing communication.

8. ***Stimulate creativity and imagination.***
   If pupils have been playing a game for some time they can be encouraged to make a similar new game for themselves or for younger brothers and sisters. Pupils frequently devise new rules to add to or to replace the basic rules to make the game more challenging to them once the basic rules have been mastered. Pupils can also be challenged to devise variations and extensions to the game. These activities call on pupils' creativity and imagination.

9. ***Serve as a source for investigational work.***
   Games can form a source for investigational work by analysing the game and answering questions such as: Is there a best move? Can the first player always win? How many possible moves are possible? Is it a fair game? What is the maximum score I can make? What would happen if ...? This can lead to looking at simpler cases first, tabulating results, making conjectures, testing hypotheses i.e. investigational work.

## Unit 1, Practice activity 1

1. Use the outline you wrote at the beginning of this unit. After you have gone through this unit and tried out some of the activities in your class, would you make changes to your original outline? Justify changes you would introduce or justify that you stay with the method you have outlined from the start.
2. Set five questions to test pupils understanding of fixing location.
3. When introducing Cartesian coordinates, would you restrict the introduction to points in the first quadrant only or would you cover all four quadrants at once? Explain and justify.
4. a) Make three different sets of coordinate sequences which, when joined in order, will give a 'picture'. One should be for lower attainers, the other for average and higher attainers in your class. Give them to your pupils to consolidate plotting of points given in coordinate form. Write an evaluative report.

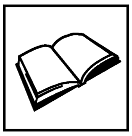
   b) Design a game for the consolidation of Cartesian coordinates. Try it out with your pupils and write an evaluative report.
5. Try out some of the suggested activities to consolidate the concept of Cartesian coordinates.

   Justify the activity you have chosen and write an evaluative report.

   Some questions you might want to answer could be: What were the strengths and weaknesses of the activity? What needs improvement? How was the reaction of the pupils? What did your learn as a teacher from the lesson? Could all pupils participate? How did you cater for the wide attainment range in your class? Were your objective(s) attained? Was the timing correct? Were you satisfied with the outcome of the activity?
6. Find out about **polar coordinates** and discuss the advantages and disadvantages as compared to Cartesian coordinates. Draw some curves given in polar coordinates.

*Present your assignment to your supervisor or study group for discussion.*

# Section B: Midpoints and distances

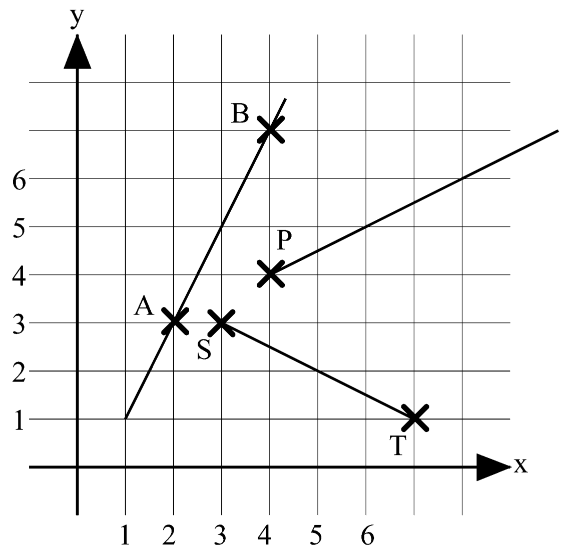


## Section B1: Midpoint of a straight line segment

Points, lines, segments are all mathematical concepts. They are abstract ideas developed within the field of mathematics and exist only in our minds. Any diagram, drawing, picture can represent the idea but is not the idea itself.

For proper understanding of the following you should distinguish between the following:

- line straight: connection through two points extending indefinitely in both directions
- line segment: part of a line between two points. A line segment has a finite length: the distance between its endpoints.
- half line or ray: a straight line extending indefinitely in one direction from a fixed point




The diagram gives a representation of the line through A and B, the half line with end point P and the line segment ST.

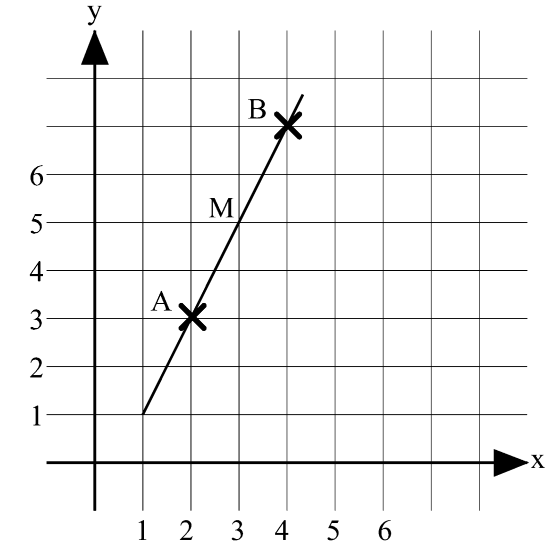
## Section B2: Midpoints - worksheets for pupils

The following is an outline of a worksheet for pupils to discover how the coordinates of the midpoint of a line segment are related to the coordinates of its endpoints. An inductive method is used: from special cases pupils are to discover a pattern and suggest a relationship. The outcome of an inductive method is always ‘probable or likely knowledge’. It is NOT a proof—only a conjecture which might or might not be correct.

Pupils are expected to work in groups and discuss their work with each other.

**Worksheet 1**

On the coordinate grid the points A(2, 3) and B(4, 7) have been plotted. The midpoint M of the line segment AB has coordinates (3, 5) as can be obtained from the grid. M is midway between the endpoints. The distance AM and BM are equal.




a) Plot the following points on a coordinate grid and write down the coordinates of the midpoint

| A | B | M |
|---|---|---|
| (0,1) | (0,7) | |
| (2,2) | (2,4) | |
| (1,4) | (7,4) | |
| (0,5) | (4,7) | |
| (0,8) | (4,0) | |
| (3,6) | (3,2) | |
| (1,1) | (5,3) | |
| (2,7) | (6,1) | |
| (4,0) | (0,8) | |
| (5,5) | (0,0) | |

b) How are the coordinates of M related to the coordinates of A and B?

c) Without plotting the points write down the coordinates of the midpoint of the line segment with endpoints
(28,4) and (8, 10) (4, 16) and (36, 12) (32, 45) and (78, 155)

d) Investigate for points with one or both coordinates negative. Make a table similar to the one in a). Does your rule still work?

| A | B | M |
|---|---|---|
| (0, ⁻1) | (0, 7) | |
| (2, 2) | (2, ⁻4) | |
| (⁻1, 4) | (⁻7, 4) | |
| (0, ⁻5) | (⁻4, 7) | |
| (0, ⁻8) | (⁻4, 0) | |
| (⁻3, 6) | (⁻3, ⁻2) | |
| (1, 1) | (5, ⁻3) | |
| (2, ⁻7) | (6, 1) | |
| (4, 0) | (0, ⁻8) | |
| (⁻5, ⁻5) | (0, 0) | |

e) Without plotting the points, write down the coordinates of the midpoint of the line segment with endpoints

(i) (⁻18, 4) and (6, ⁻10)

(ii) (24, ⁻60) and (⁻34, ⁻12)

(iii) (36, 45) and (⁻78, ⁻135)

f) (i) The midpoint of AB is (2, 3) and point A is at (1, 1). What are the coordinates of B?

(ii) The midpoint of AB is (⁻2, 3) and point A is at (1, 1). What are the coordinates of B?

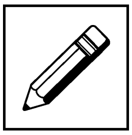
(iii) The midpoint of AB is (2, ⁻3) and point A is at (1, 1). What are the coordinates of B?

(iv) The midpoint of AB is (⁻2, ⁻3) and point A is at (1, 1). What are the coordinates of B?

(v) Tabulate your results.

| A | M | B |
|---|---|---|
| (1, 1) | (2, 3) | |
| (1, 1) | $(^{-}2, 3)$ | |
| (1, 1) | $(2, ^{-}3)$ | |
| (1, 1) | $(^{-}2, ^{-}3)$ | |
| | | |
| | | |
| | | |
| | | |
| | | |
| | | |

Can you find a rule? Check your rule for some more cases given coordinates of the midpoint and point A.



## Self mark exercise 3

1. Work through the pupils' worksheet 1.

*Check your answers at the end of this unit.*

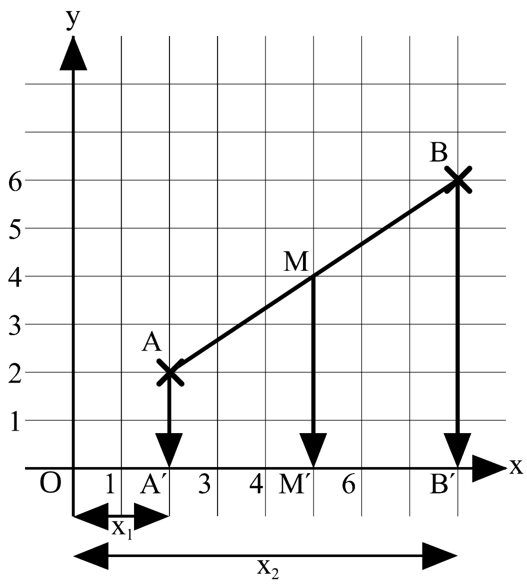
**Worksheet 2**

Worksheet for higher attainers. Deductive discovery method is used: from previous knowledge new knowledge is developed. Deductive methods lead to ‘sure’ knowledge, it proves the result.

Pupils are expected to work in groups and discuss their work with each other.

a) The diagram illustrates the line segment AB with midpoint M. Let us take A and B as some general points and derive the coordinates of the midpoint M.

   Let A($x_1, y_1$) and B($x_2, y_2$)




   The $x$-coordinate of A, $x_1$, is represented in the diagram by OA′.
   Hence $x_1$= OA′

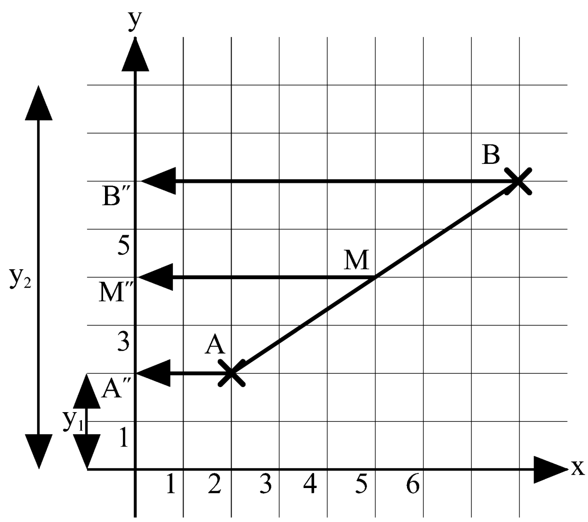
   Complete the following to find the $x$-coordinate of M, the midpoint of AB.
   A′B′ = OB′ – OA′ = $x_2$ – ....

   A′M′ = $\frac{1}{2}$ A′B′ = $\frac{1}{2}(x_2 - ...)$

   OM′ = $x_M$ = OA′ + A′M′ = $x_1$ + $\frac{1}{2}(x_2 - ...)$

   (expand and simplify) = ................
   Hence the $x$-coordinate of the point M: $x_M$ = .........

b) Repeat the above but now for the $y$-coordinates. Use the following diagram and find an expression for OM″ = $y_M$, the $y$-coordinate of the midpoint.




Complete the following to find the $y$-coordinate of M, the midpoint of AB.

A″B″ = OB″ - OA″ = ..... ....

A″M″ = $\frac{1}{2}$ A″B″ = ......

OM″ = $y_M$ = OA″ + A″M″ = ........... (expand and simplify)

= ................

Hence the $y$-coordinate of the point M: $y_M$ = .........

c) Summarise your finding:
If A($x_1$, $y_1$) and B($x_2$, $y_2$) are the coordinates of the endpoints of line segment AB then the coordinates of the midpoint M of AB are: (.... , ....)

d) Calculate the coordinates of the midpoint of the line segments with, as endpoints, the points with the following coordinates:

(i) (0, ⁻3) and (5, 0)

(ii) (⁻1$\frac{1}{2}$ , 4) and (0, ⁻2$\frac{1}{2}$ )

(iii) (3.4, ⁻6.7) and (⁻8.3,17.2)

e) The coordinates of the midpoint of the line segment with coordinates of its endpoints ($x$, ⁻5) and (0, ⁻3) are (2, $y$). Calculate the value of $x$ and $y$.

f) The coordinates of the midpoint of the line segment with coordinates of its endpoints ($x$, 2.5) and (⁻6, 4) are (⁻3$\frac{1}{2}$ , $y$).

Calculate the value of $x$ and $y$.

g) The coordinates of the midpoint of the line segment with coordinates of its endpoints (⁻3.4, 2.5) and (⁻6, 4) are ($x$, $y$). Calculate the value of $x$ and $y$.

## Self mark exercise 4

1. Work through the pupils' worksheet 2.
2. Using a similar deductive method as in worksheet 2 show that the coordinates of the point P dividing the line segment AB into the ratio AP : PB = 1 : 3 are given by

   P$\left(\frac{3x_1 + x_2}{4}, \frac{3y_1 + y_2}{4}\right)$, where the coordinates of the endpoints of the line segment are A($x_1$, $y_1$) and B($x_2$ ,$y_2$).
3. Using a similar deductive method as in question 2 show that the coordinates of the point P dividing the line segment AB into the ratio AP : PB = $p$ : $q$ are given by

   P$\left(\frac{qx_1 + px_2}{p+q}, \frac{qy_1+py_2}{p+q}\right)$,where the coordinates of the endpoints of the line segment are A($x_1$, $y_1$) and B($x_2$ ,$y_2$).

*Check your answers at the end of this unit.*

## Unit 1, Practice activity 2

1. You learned to distinguish between line segment, half line (or ray) and line. Do you feel these differences are important to pupils, i.e. would you introduce the concepts to pupils or not? Justify.
2. You worked through the suggested worksheet W1 and W2 for pupils to discover how the coordinates of the midpoint of the line segment is related to the coordinates of its end points. Will it be appropriate for use in your class? Justify your answer.

   Does it need improvement? Does it cater for all pupils or is a version, more guided, for lower attainers needed? If yes, develop such a worksheet.
3. Try out the (improved) version(s) of the worksheet in your class and write an evaluative report.
4. Both worksheets need to be supported by consolidation work. Write for both worksheets 10 consolidation questions. Justify why you included each question in the exercise and show the working you expect from pupils working the exercise.

*Present your assignment to your supervisor or study group for discussion.*

## Section B3: Distance between two points with coordinates ($x_1$, $y_1$) and ($x_2$, $y_2$)

The distance between two points given their coordinates is an application of the Pythagorean theorem, and should be introduced after the Pythagorean theorem has been covered by the pupils.

The assumed and prerequisite knowledge for this section is therefore the Pythagorean theorem.

### Self mark exercise 5

1. Give a concise formulation of the Pythagorean theorem in terms of length of the sides of a right-angled triangle.
2. Give a concise formulation of the Pythagorean theorem in terms of the area enclosed by squares on the sides of a right-angled triangle.
3. a) If $d^2 = 9$, find the value(s) of $d$.

   b) If $d = \sqrt{9}$ , find the value(s) of $d$.

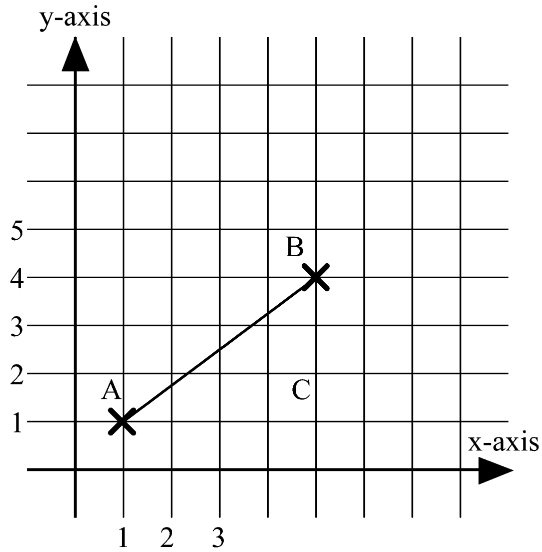
*Check your answers at the end of this unit.*

As pupils are assumed to be familiar with the Pythagorean theorem, the 'distance formula' can be developed through (i) teacher guided class discussion (ii) guided worksheet.

An outline of a worksheet follows here. Pupils are expected to work in groups and discuss their work with each other.

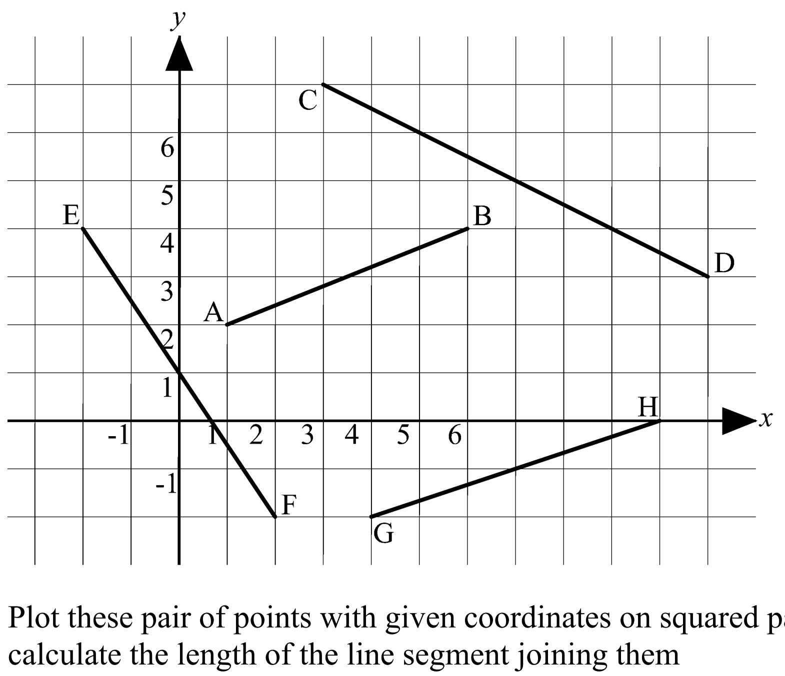
## Worksheet 3

a) On a map two places A and B are situated at (1, 1) and (5, 4) respectively (distances in km). A road is to be constructed connecting the two places directly. What will be the length of the road?




AB is the hypotenuse of the right angled triangle ACB.
What is the length of AC?
What is the length of BC?
Apply the Pythagorean theorem to find the distance AB.

b) Calculate the lengths AB, CD, EF and GH, correct to 1 decimal place.

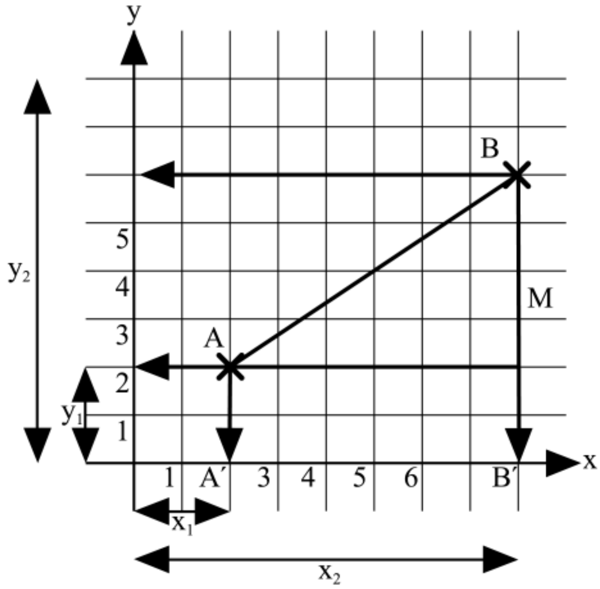



c) Plot these pair of points with given coordinates on squared paper and calculate the length of the line segment joining them

(i) O(0, 0) and P(8, 6) (ii) Q(1, 2) and R(-4, 7)

(iii) S($^-$6, 8) and T($^-$1, $^-$1)

d) Let $A(x_1, y_1)$ and $B(x_2, y_2)$




The $x$-coordinate of A, $x_1$, is represented in the diagram by OA′.
Hence $x_1$ = OA′

AC = A′B′ = OB′ - OA′ = $x_2$ – .... (i)

BC = ...... (ii)

Applying the Pythagorean theorem in triangle ACB: $AB^2 = AC^2 + BC^2$

Substitute for AC and BC the expressions obtained in (i) and (ii) respectively

$AB^2 = (x_2 - ....)^2 + ...........$

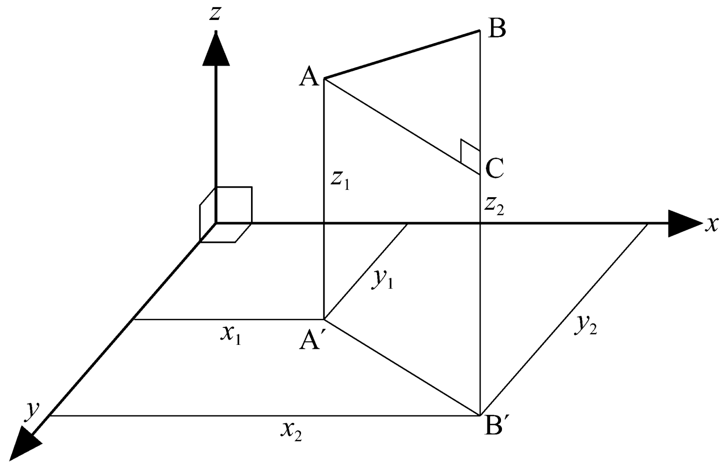
AB = ..... (or AB = - ...., not acceptable as distances are positive).

In summary if the coordinates of two points are $A(x_1, y_1)$ and $B(x_2, y_2)$ then the distance between the two points is ......

The notation used in several books for the distance between two points A and B is |AB|, meaning the **modulus** or length of the line segment with end points A and B. Since "modulus" sounds a lot like the word modulo (as in q mod 4 = 1), it's use for the length of a line segment is not recommended for secondary students.

## Self mark exercise 6

1. Work through pupils' worksheet 3.
2. The outcome of worksheet 3 is the distance formula for the distance between two points A($x_1$, $y_1$) and B($x_2$, $y_2$).
   |AB| = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
   Explain why this can also be written as |AB| = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
3. a) In 3 dimensions you will need three coordinates to fixed the position of a point. What is the distance between two points A($x_1$, $y_1$, $z_1$) and B($x_2$, $y_2$, $z_2$)? First find A´B´ (= AC), next apply the Pythagorean theorem in triangle ACB to find AB.




   b) What are now the coordinates of the midpoint M of the segment AB?
4. If A(0, 0) and B(b, 0) find the coordinates of the point(s) C such that
   a) triangle ABC is isosceles
   b) triangle ABC is equilateral

*Check your answers at the end of this unit.*

## Unit 1, Practice activity 3

1. In this unit you met with two distinct formulations of the Pythagorean theorem: one in terms of areas of squares on the sides of the right-angled triangle and the other in terms of lengths of the sides.
   a) Which of these two do you use in setting activities that lead the pupils to the discovery of the theorem?
   b) Describe the method you use to move from the Pythagorean theorem form discovered by the pupils to the other form.
2. Analytical or coordinate geometry is a method that can be used to prove geometrical properties that might have been obtained in a geometrical way (using symmetry or congruency).
   a) Design a worksheet for pupils to prove using coordinate geometry the following geometrically known properties:
      (i) the diagonals in a rectangle are equal in length
      (ii) the diagonals in a rectangle bisect each other
   b) Try out the worksheet in your classroom
   c) Discuss with your pupils which method has their preference: the geometrical or the analytical way to show the property
   d) Write an evaluative report on the lesson, including reactions of pupils.

*Present your assignment to your supervisor or study group for discussion.*

## Summary

This unit introduced some modern classroom techniques for starting your pupils in analytic geometry. Likely these methods are different from ones you were taught as a child. For one thing, they employed the relatively "sloppy" inductive method; rigorous proof was left out, or was set aside for just the higher attainers in your class. For another thing, did you notice that Euclid's geometry is not taught in this course at all? In effect, Descartes has supplanted Euclid. Traditionalists bemoan Euclid's passing, but the change simply reflects the relative power of Cartesian methods for solving geometrical puzzles in everyday life. In like vein, students at this age have a better grasp of inductive reasoning. For purists who say that the more formal logic of deduction and proof is necessary for endeavours like law, the facetious answer is to teach it in law school! But more realistically, it is better used as a learning medium when most students are a few years older.

## Unit 1: Answers to self mark exercises

**Self mark exercise 1**

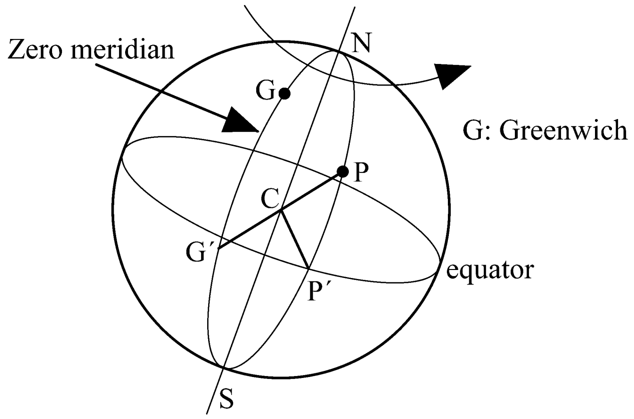
1. There are various ways to describe the position of a point accurately: using distance(s) and/or direction(s) from (a) fixed point(s).

   e.g. a point P in a plane can be fixed by any of the following

   - the Cartesian coordinates of P with respect to a fixed point
   - distance and direction (bearing or angle made with a fixed line e.g. $x$-axis) from a fixed given point (polar coordinates)
   - directions from two fixed given points
   - distances from three given fixed points (distances from two points does not uniquely fix the position/it allows for two possibilities).

2. Latitude - angular distance of a point on the earth's surface measured from the equator along the meridian passing through that point

   Longitude - angular distance of a point on the earth's surface measured along the equator between the prime (zero or Greenwich) meridian and the meridian through the point




   The latitude of the place P is the size of angle PCP′, where C is the centre of the earth and P and P′ are on the meridian through P, with P′ on the equator.

   The longitude of P is the size of angle G′CP′

**Self mark exercise 2**

1. P1 bottom far right/bottom far left  P2 B(6, 4)
2. P1 analysis  P2 application/analysis

**Self mark exercise 3**

1. Worksheet W1

   a) (0, 4); (2, 3); (4, 4); (2, 6); (2, 4) ; (3, 4); (3, 2); (4, 4); (2, 4); (2.5, 2.5)

   b) $x$-coordinate of the midpoint is the average of the $x$-coordinates of the endpoints. $y$-coordinate of the midpoint is the average of the $y$-coordinates of the endpoints.

   c) (18, 7); (20, 14); (55, 100)

   d) $(0, 3)$; $(2, {}^{-}1)$; $({}^{-}4, 4)$; $({}^{-}2, 1)$; $(-2, {}^{-}4)$; $({}^{-}3, 2)$; $(3, {}^{-}1)$; $(4, {}^{-}3)$; $(2, {}^{-}4)$; $({}^{-}2.5, {}^{-}2.5)$

   e) $({}^{-}6, {}^{-}3)$; $({}^{-}5, {}^{-}36)$; $({}^{-}21, {}^{-}45)$

   f) (i) $(3, 5)$; (ii) $({}^{-}5, 5)$; (iii) $(3, {}^{-}7)$; (iv) $({}^{-}5, {}^{-}7)$

   g) $(3, 5)$; $({}^{-}5, 5)$, $(3, {}^{-}7)$, $({}^{-}5, {}^{-}7)$

**Self mark exercise 4**

1. Worksheet W2

   a) $A'B' = x_2 - x_1 \quad A'M' = \frac{1}{2}(x_2 - x_1)$

   $OM' = x_1 + \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(x_2 + x_1) \quad x_M = \frac{1}{2}(x_1 + x_2)$

   b) As (a) replacing $x$ by $y$.

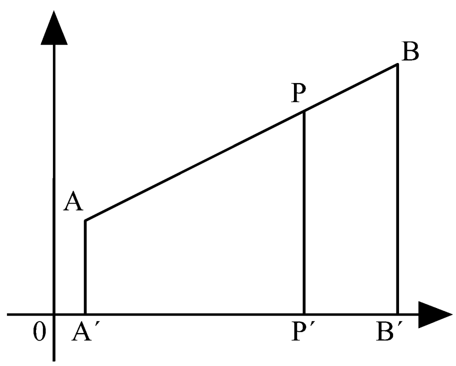
   c) $M\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(x_1 + x_2)\right)$

   d) $(2.5, {}^{-}1.5)$; $\left({}^{-}\frac{3}{4}, {}^{-}\frac{3}{4}\right)$; $({}^{-}2.45, 5.25)$

   e) $x = 4, y = {}^{-}4$ f) $x = {}^{-}1, y = 3.25$ g) $x = {}^{-}4.7, y = 3.25$

3. In the diagram $AP : PB = A'P' : P'B' = p : q$

   $A'P' = \frac{p}{p+q} A'B' = \frac{p}{p+q}(x_2 - x_1)$



$$OP' = x_p = x_1 + \frac{p}{p+q}(x_2 - x_1) =$$

$$\frac{(p+q)\,x_1 + p(x_2 - x_1)}{p+q} = \frac{qx_1 + px_2}{p+q}$$

Similarly for the *y*-coordinate of the point P.

**Self mark exercise 5**

1. In a right angled triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two right angled sides.
2. The area of the square on the hypotenuse of a right-angled triangle is equal to the areas of the squares on the two right-angled sides of the triangle.
3. a) $^-3$, 3  b) 3 (NOT $^-3$)

**Self mark exercise 6**

1. Worksheet 3

   (a) 5 km; AC = 4, BC = 3, AB = 5

   (b) 5.4; 8.9; 7.2; 6.3

   (c) (i) 10  (ii) $5\sqrt{2}$  (iii) $\sqrt{106}$

   (d) $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

2. $(x_2 - x_1)^2 = x_2^2 - 2x_1x_2 + x_1^2$ and $(x_1 - x_2)^2 = x_1^2 - 2x_1x_2 + x_2^2$ hence

   $(x_2 - x_1)^2 = (x_1 - x_2)^2$.

   Similarly for the expression in the *y*-coordinates, hence:

   $$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

3. a. $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

   b. $\left(\frac{x_1 + \ + x_2}{2}, \frac{y_1 + \ + y_2}{2}, \frac{z_1 + \ + z_2}{2}\right)$

4. a. $(\frac{b}{2}, c)$  b. $(\frac{b}{2}, \frac{b}{2}\sqrt{3})$ *or* $(\frac{b}{2}, \frac{^-b}{2}\sqrt{3})$

# Unit 2: Coordinate geometry II

## Introduction to Unit 2

This unit looks at 'straight lines'. Some people query the word 'straight' as in their definition of a line a line, is always straight. If it is not straight it is not a line. The concept "straight line" seems obvious to many a person until asked to define exactly what is meant by it. Attempts to define a straight line include those by:

Plato (380 BC): "that of which the middle covers the ends" (an expression of the view from the eyes placed at either end, looking along the line).

Euclid (300 BC): "that which lies evenly with points on itself."

Archimedes (225 BC): "of all lines having the same extremities the straight line is the shortest". This is the source of the common definition: the shortest distance between two points.

This however does not solve our problems completely, as commonly a point is defined as "the intersection of two lines" i.e., a circular definition!

## Purpose of Unit 2

The purpose of this unit is to revise your knowledge on gradients of straight lines, relations between gradients of lines that are parallel or perpendicular to each other and on equations of straight lines. The emphasis is on how you can present the concepts to the pupils in your class.

Geometrical properties are investigated using coordinate geometry.

## Objectives

When you have completed this unit you should be able to:

- set learning activities to your pupils to cover the concepts (i) gradient or slope (ii) $y$-intercept (iii) $x$-intercept
- identify straight lines with negative, zero, positive and no gradient
- relate gradients with angles the line makes with the positive $x$-axis
- set learning activities to your pupils to discover that (i) parallel lines have equal gradients (ii) lines with equal gradients are parallel (iii) the gradients of perpendicular lines have as product $^{-}1$, provided no horizontal /vertical line is involved (iv) if the product of the gradients of two lines is $^{-}1$ the lines are perpendicular
- prove that (i) if two lines (not horizontal or vertical) are perpendicular the product of their gradients is $^{-}1$ (ii) if the product of the gradients of two lines is $^{-}1$ the lines are perpendicular
- plot graphs of straight lines given their equation
- use coordinate geometry in investigating geometrical properties